# GENERAL SYNCHRONIZATION OF COUPLED PAIR OF CHAOTIC ONE-DIMENSIONAL GAUSSIAN MAPS

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## Abstract

In this paper we review some recent ideas of synchronization theory. We apply this theory to study the different synchronization aspects of uni-directionally coupled pair of chaotic one-dimensional Gaussian maps.

## 1. Introduction

The first observation of a synchronization phenomenon is attributed to C. Huygens in 1673 [10] during his experiments for developing improved pendulum clocks. Two clocks hanging on the same beam of his room were found to oscillate with exactly the same frequency and opposite phase due to the coupling in terms of the almost imperceptible oscillations of the beam generated by the clocks. In recent years, the synchronization of coupled chaotic systems has become an area of active research. The motivation for these investigations derived from possible applications of this phenomenon to secure communications [7], the long-term prediction of chaotic systems [18], controlling chaos [12], the model verification of non-linear dynamics [4], or the estimation of model parameters [16].

The most important feature of non-linear systems exhibiting chaotic motion is extreme sensitivity to initial conditions. This feature, known as the "butterfly effect", would seem to defy synchronization among dynamical variables in coupled chaotic systems. Nonetheless, coupled systems with certain properties of symmetry may exhibit synchronized chaotic motion [13]. Most frequently, a situation is studied where the complete system consists of coupled identical subsystems. Many different examples of this type have been introduced [6], [17] and [18]. In these cases, the synchronization is easy to detect. It appears as an actual equality of the corresponding variables of the coupled systems as they evolve in time. Geometrically, this implies a collapse of the overall evolution onto the identity hyperplane in the full phase space. As suggested in Ref. [15], we refer to this type of synchronization as an identical synchronization (IS).

A more complicated situation arises when coupled non-identical chaotic systems are investigated. For essentially different chaotic systems, the phase space does not contain any trivial invariant manifolds from which one can expect a collapse of the overall evolution. The central questions in this case are (i) how to generalize a mathematical definition of chaotic synchronization for such systems and (ii) how to detect it in a real experimental situation. Recently, two approaches have been suggested in order to answer these questions. One of them [22] uses the concept of an analytical signal and introduces an instantaneous phase and amplitude for the chaotic process. The synchronization appears as locking of the phases of coupled systems, while the amplitudes remain uncorrelated. This type of synchronization is identified as a phase Synchronization. Another approach [2] is based on the concept of the functional relationship

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between the variables of the coupled subsystems. This approach can lead to the existence of a function that maps (asymptotically for $t \to \infty$) states of the drive system to states of the response system. This type of synchronization is called generalized synchronization (GS). In this case the chaotic dynamics of the response system can be predicted from the drive system. There are some other approaches, which are discussed elsewhere, [13].

In this study, first, we briefly describe the main ideas defining the concept of GS. With the help of PHASER software [14] we show that for system of chaotic Gaussian maps GS may appear in two different synchronization states, referred to as weak synchronization (WS) and strong synchronization (SS). Lyapunov exponents of linearized chaotic system are the most common tools for study the stability of the synchronization manifold. We use this tool to show the (in)stability of the synchronization manifold for different values of coupling strength parameter, which exists in the system of Gaussian maps. We end up this study with the conclusions of our study.

## 2. Definitions and Geometry of Synchronization

Consider a coupled pair of one-dimensional Gaussian maps:

$$x_{n+1} = f(x_n) \quad (1)$$

$$y_{n+1} = f(x_n) + c[f(x_n) - f(y_n)], \quad (2)$$

for $f(x_n) = \exp[-a^2(x_n - b)^2]$, plus a coupling term $c[f(x_n) - f(y_n)]$, where $c$ is the scalar coupling strength. The coupling acts like negative feedback. If we set *c=0, a=3.5* and *b=0.5* we get independent chaotic systems. The dynamical variables ( $x_n$ and $y_n$ ) in each system remain uncorrelated with each other. Now if we take $c \in (0,1)$, we see a new behavior set in. In this case, as we will see, for different values of $c \neq 0$ in this interval $|x_n - y_n| \to 0$ for large *n*. we now have a set of synchronized, chaotic systems. The dynamical variable in one system is equal to its counterpart in the other. More importantly, we can get an idea of what the geometry of the synchronous attractor looks like in phase space. Typical figures displaying synchronous systems usually look like Figure 1, just a 45° line showing that a variable from one system equals its counterpart in the other system of all time. Systems (1) and (2) are called derive and response systems, respectively.

Two identical systems are in IS if the attractor lies on a hyperplane which its dimension is strictly less than the full phase space dimension. If two different systems are coupled identical synchronization is in general not possible, but other types of synchronization may be observed. For chaotic systems Afraimovich *et al.* [2] gave the first definition for what was later called GS by Rulkov *et al.* [23]. In this definition Afraimovich *et al.* called two systems synchronized if in the limit $t \to \infty$ ($n \to \infty$) a homeomorphic function exists mapping states of one system to states of the other. Later the assumption of a homomorphism was relaxed and two systems are said to be in synchrony if there states **x** and **y** are asymptotically related by some function **H** so that $\|\mathbf{H}(\mathbf{x}(t)) - \mathbf{y}(t)\| \to 0$ for $t \to \infty$. This definition of GS was used in Refs. [9] and [15].

For non-identical driving and response systems, the map differs from identity, which complicates the detection of GS. To recognize GS in a real experimental situation, Rulkov *et al.* [23] suggested a practical algorithm based on the assumption that **H** is a smooth (differentiable) map. The algorithm was tested on artificially constructed examples with a prior known map **H**. Subsequent progress of GS theory was achieved in recent publications [1], [15], [19], [20], [21] and [24]. Depending on the properties of the map **H**, two different types of GS where discovered [19], namely, SS and WS, which are characterized by a smooth and a non-smooth map **H**, respectively.

Now in the following section we will detect different synchronization of coupled Gaussian maps (1) and (2).

## 3. Synchronization of Gaussian Maps

Consider again the system of chaotic Gaussian maps:

$$x_{n+1} = \exp[-a^2(x_n - b)^2] \quad (3)$$

$$y_{n+1} = (1-c)\exp[-a^2(y_n - b)^2] + c\exp[-a^2(x_n - b)^2] \quad (4)$$

As deriving and response subsystems, respectively with an auxiliary response subsystem

$$z_{n+1} = (1-c)\exp[-a^2(z_n - b)^2] + c\exp[-a^2(x_n - b)^2] \quad (5)$$

Note that the auxiliary response subsystem (5) is identical with the response subsystem (4). We emphasize that subsystem (5) does not influence the dynamics of the original response and deriving

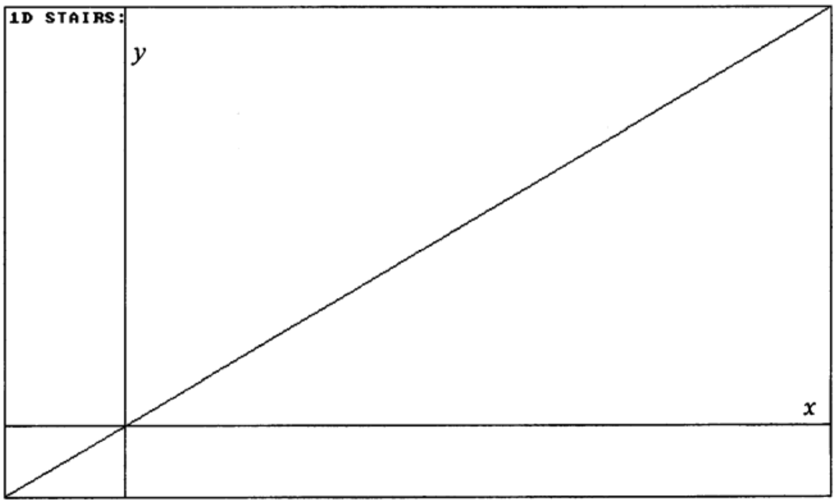


**Figure 1.** Typical figure of synchronization between two systems.

subsystems (4) and (3). It serves only to detect the WS of the system.

To illustrate the experimental observation of GS for this system we have used the software PHASER [14]. This software enables us to solve and illustrate the (chaotic) solutions of above system for different values of c in different two-dimensional coordinates $x_1$, $x_2$ and $x_3$ versus $n$ ($t$). As we will see in the next section, these experimental observations are agree with other results using the tools for detecting and analyzing the synchronization of the chaotic systems.

We have run the PHASER to solve the system (3-5) with different initial conditions and coupling strength $c$. The results are shown in Figure 2. For values $0 < c < 0.3$ there is no synchronization between ($x_n$, $y_n$) and ($y_n$, $z_n$) (Fig. 2a,b), for various values $0.42 \le c \le 0.662$ synchronization occurs between $x_n$ and $y_n$ but the smooth identity manifold $y = x$ is unstable, i.e., we may have $y_n = x_n$ for some $n$ (for while) but later on for some bigger $n$, this identity will break down.

Indeed, as we will see in the next section, this is the case of WS. Thus GS in the form of WS is observed for this identical system (Fig. 2c,d). With the increase of $c > 0.662$ subsystems (3) and (4) have an invariant manifold $y = x$ and, hence admit IS. This case is interesting, since it provides a simple criterion for SS. Indeed, the variables of the response and deriving subsystems are related by the identity map $y$=$x$, which obviously is smooth. Hence, SS can be simply detected as IS between the deriving and response subsystems. Figures 2(a)-2(f) show the solutions $x_n$, $y_n$ and $z_n$ of the system versus $n$ ($t$) at $a$ = 3.5, $b$ = 0.5 and for various values of parameter $c$.

## 4. Stability of the Synchronization Manifold

Several methods have been discussed to study the stability of the synchronization manifold [3,5,11,17, 19]. Using the Lyapunov exponents of linearized chaotic system is the most common method to study the stability of the synchronization manifold. In general, consider $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x_n})$, $\mathbf{y_{n+1}} = \mathbf{g}(\mathbf{y_n}, \mathbf{x_n})$ and $\mathbf{z_{n+1}} = \mathbf{g}(\mathbf{z_n}, \mathbf{x_n})$ as derive, response and auxiliary response subsystems respectively. Then IS occurs if the dynamical system describes the evolution of the difference $\mathbf{e_n} = \mathbf{y_n} - \mathbf{x_n}$

$$\mathbf{e_{n+1}} = \mathbf{f}(\mathbf{x_n}) - \mathbf{g}(\mathbf{y_n}, \mathbf{x_n})$$

Possesses a stable fixed point at the origin $\mathbf{e} = \mathbf{0}$. In some cases this can be proved using stability analysis of the linearized system for small **e**,

$$\mathbf{e_{n+1}} = D\mathbf{f}_{\mathbf{x_n}}\mathbf{e_n} \tag{6}$$

or using Lyapunov functions. In general, however, the stability has to be checked numerically by computing so-called transversal or conditional Lyapunov exponents (CLEs) using the linearized equation (6). IS occurs if all transversal Lyapunov exponents of the system $\mathbf{x_{n+1}} = \mathbf{f}(\mathbf{x_n})$ are negative, otherwise the synchronization manifold of the system ($\mathbf{y_n} = \mathbf{x_n}$) is unstable. In this case similar analogous can be use for WS; i.e. if the CLEs of linearized equation

$$\mathbf{e}'_{\mathbf{n+1}} = D\mathbf{g}_{\mathbf{y_n}}\mathbf{e}'_{\mathbf{n}} \tag{7}$$

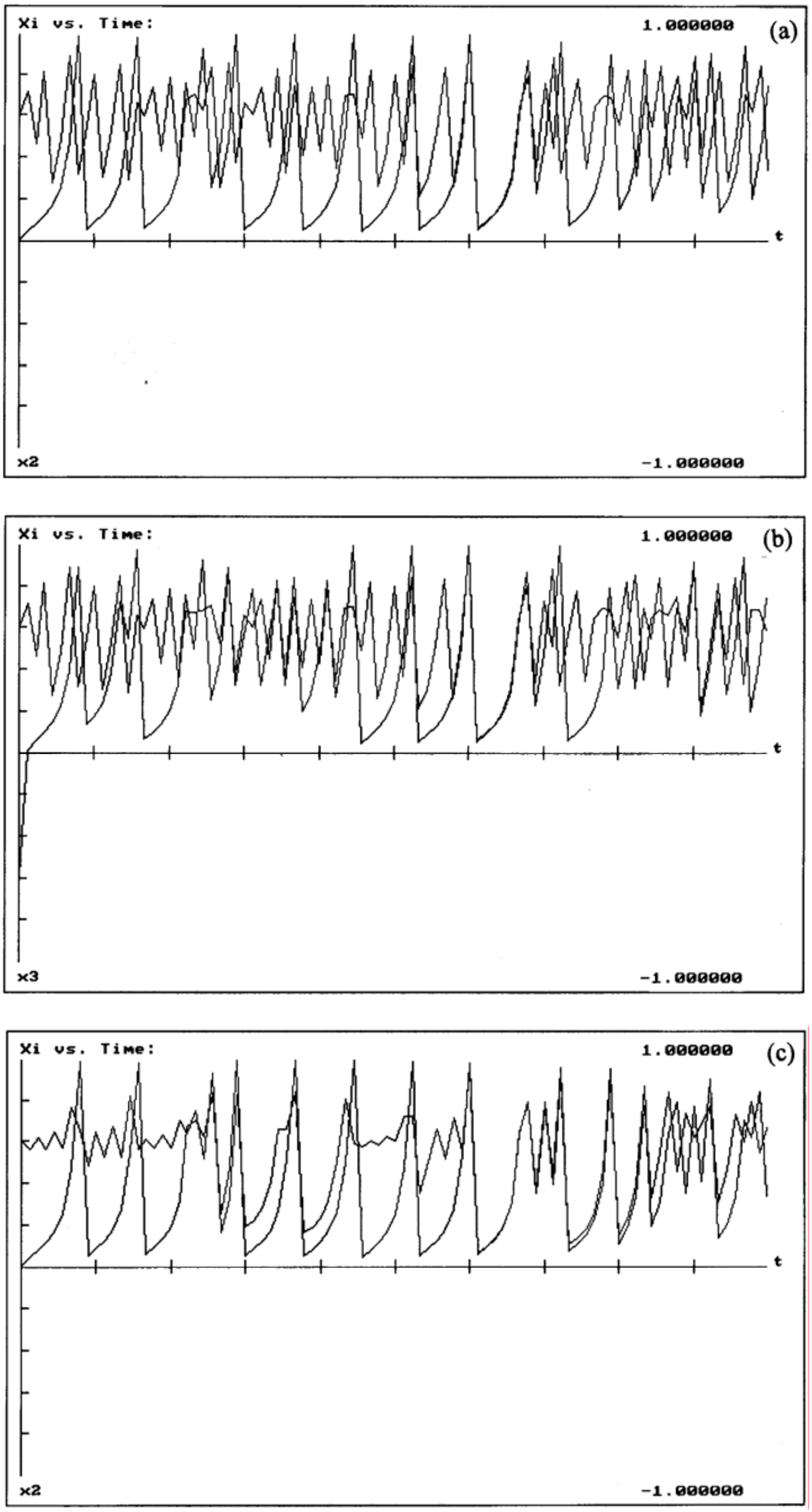


**Figure 2.** Solutions of coupled Gaussian maps $x_n$, $y_n$ and $z_n$ (x1, x2, and x3) *vs* $n$ (time) for various values of the coupling strength $c$. In all these figures $x_n$, $y_n$ and $z_n$ are started from 0, 0.6 and –0.6, respectively. (a) and (b) $c$=0.2 there is no synchronization. (c) and (d) c=0.42, WS. (e) and (f) $c$=0.7, SS.

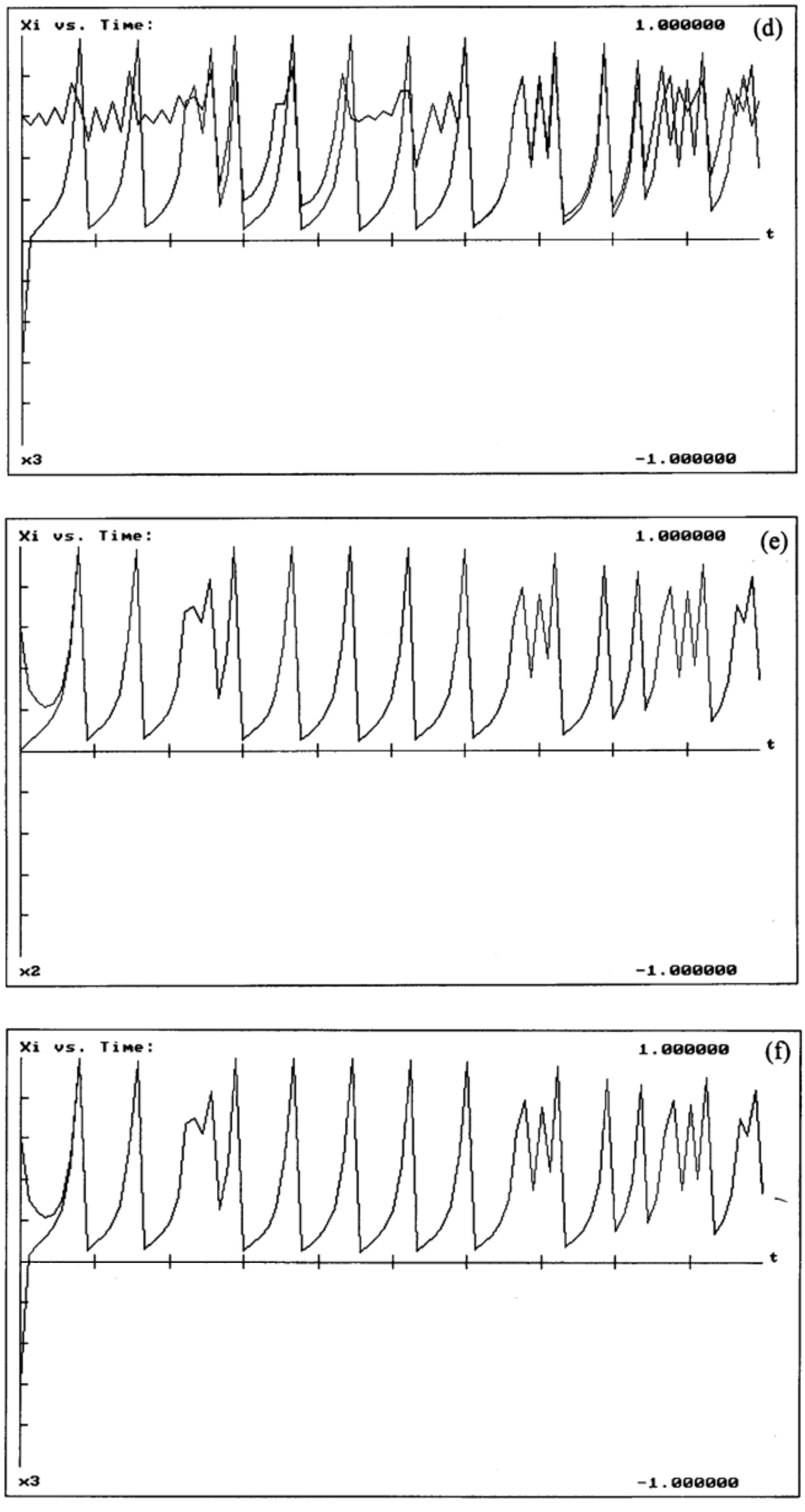


**Figure 2.** Continued.

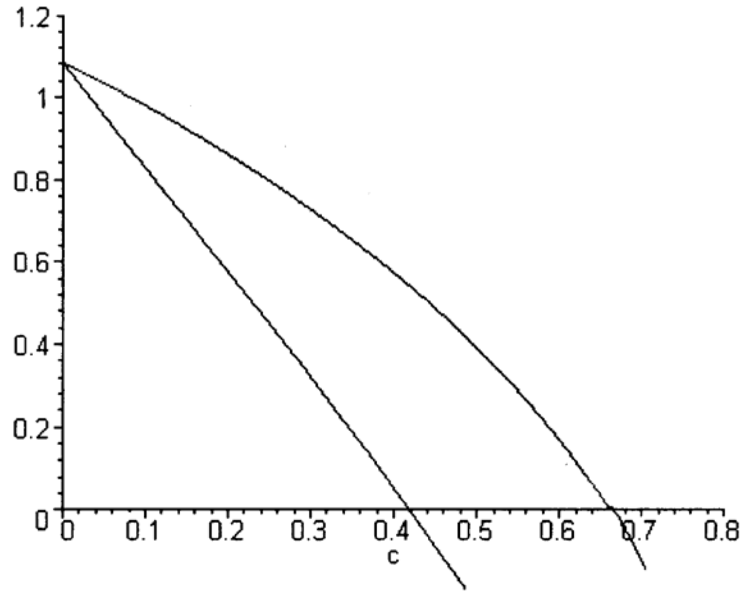


**Figure 3.** CLEs $\lambda^R$ and transverse Laypunov exponents $\lambda^I$. Unsynchronized state for $c \in (0,0.4)$, WS for $c \in (0.42,0.662)$ and SS for $c > 0.662$.

are negative, then the synchronization manifold of the response and auxiliary response subsystems ($\mathbf{z_n} = \mathbf{y_n}$) is stable. So SS occurs if all transversal and conditional Lyapunov exponents are negative. Note that here $\mathbf{e'_n} = \mathbf{z_n} - \mathbf{y_n}$.

Now going back to the system (3-5) we can use both stability analyses of the linearized system and Lyapunov exponents to discuss the stability analysis of its synchronization manifold. To illustrate this stability analysis we consider the fixed point $(\bar{x}_n, \bar{y}_n) =$ $(0.678082, 0.678082)$ of the coupled subsystems (3-4) that is located on the synchronization manifold ($y_n = x_n$). The stability features of this fixed point are given by the $\lambda_1 = f'(\bar{x}_n) = -2.458479$ and $\lambda_2 = (1-c) f'(\bar{y}_n)$ $= (1-c)(-2.958479)$ of the Jacobian matrix of (3-4) at $(\bar{x}_n, \bar{y}_n)$. As we can see $|\lambda_1| > 1$, which describes the instability within the synchronization manifold and does not depend on the coupling. The second eigenvalue, however, reflects the (in)stability and depends on *c*. For example if $c > 0.662$ the stability criterion $|\lambda_2| < 1$ holds; i.e. in this case the corresponding eigenvalue(s) of system (6) is less then one and we have stable synchronization manifold for $y_n = x_n$. For more precise study of GS in the form of WS and SS, we have used two different Lyapunov exponents [3], namely, the CLE

$$\lambda^R = \ln(1-c) + \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln|f'(y_n)| \tag{8}$$

defining the stability of the invariant manifold $z_n = y_n$, and the transverse Lyapunov exponent of the identity manifold $y_n = x_n$

$$\lambda^I = \ln(1-c) + \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln|f'(x_n)| \tag{9}$$

As discussed in [8] if $(\bar{x}_n, \bar{y}_n)$ is a fixed point of the system then two limits in (8) and (9) are simply the logarithms of the moduli of eigenvalue $\lambda_1$ and $\lambda_2$. In this case if we take $(\bar{x}_n, \bar{y}_n)$ =(0.678082, 0.678082), the $\lambda^I$ and $\lambda^R$ can easily be computed for different values of *c*. Indeed, the dependence of these exponents on *c* is shown in Figure 3. $\lambda^R$ and $\lambda^I$ become zero at two values $c_0 = 0.42$ and $c_1 = 0.662$, respectively. So WS occurs for values of $c_0 < c < c_1$ on which $\lambda^I > 0$ and $\lambda^R < 0$. For $c > 0.662$ both $\lambda^I$ and $\lambda^R$ are negative, so we have SS. Note that for the smaller *c* like $c = 0.1$ there is no stable synchronized manifold.

## 5. Conclusion

In the last few years, chaos synchronization has become one of the most intensely studied topics in nonlinear dynamics. This phenomenon is typical for couple chaotic systems. It appears, when under the action of the deriving system, the response system forgets its initial conditions and becomes an asymptotically stable system. i.e., when any initial conditions in the response lead to the same asymptotic dynamics. Experimentally, this, as we have seen, means that an ensemble of identical response subsystems driven with the same chaotic signal should exhibit identical asymptotic behavior. We have also seen that, to detect GS, one requires an ensemble of identical response subsystems. This ensemble should consist of at least two identical subsystems, namely, the original and the auxiliary response subsystems.

Conditional Laypunov exponents, is an alternative tool which can be used to detect and analyze the GS of chaos. We used these parameters to define the existence and the properties of GS in Gaussian maps.

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