FUZZY IDEALS OF NEAR-RINGS WITH INTERVAL VALUED MEMBERSHIP FUNCTIONS

B. Davvaz*

Department of Mathematics, Yazd University, Yazd, Islamic Republic of Iran

Abstract

In this paper, for a complete lattice \( L \), we introduce interval-valued \( L \)-fuzzy ideal (prime ideal) of a near-ring which is an extended notion of fuzzy ideal (prime ideal) of a near-ring. Some characterization and properties are discussed.

1. Introduction

Zadeh in [19] introduced the concept of a fuzzy subset of a non-empty set \( X \) as a function from \( X \) to [0,1]. Goguen in [10] generalized the fuzzy subset of \( X \), to \( L \)-fuzzy subset, as a function from \( X \) to a lattice \( L \).

Since Rosenfeld [18] in 1971 introduced the concept of fuzzy subgroups following Zadeh, fuzzy algebra theory has been developed by many researchers. Liu [12] defined the fuzzy ideals of a ring and discussed the operations on fuzzy ideals. Mukherjee and Sen [16], Malik and Mordeson [16], Mashinchi and Zahedi [14], Zahedi [21], showed the meaning of the fuzzy prime ideals and its nature. The notion of fuzzy ideals and its properties were applied to various areas: distributive lattice [2], BCK-algebra [17], hyperrings [6,8], near-rings [1,11], hypernear-rings [7].

In 1975, Zadeh [20] introduced the concept of interval-valued fuzzy subsets (in short written by i-v fuzzy sets), where the values of the membership functions are intervals of numbers instead of the numbers. In [4], Biswas defined interval-valued fuzzy subgroups of the same nature of Rosenfeld’s fuzzy subgroups.

In this paper, for a complete lattice \( L \), we define Interval-valued \( L \)-fuzzy ideals (prime ideals) of a near-ring, and we obtain an exact analogue of fuzzy ideals. In particular, we show there exists a one-to-one correspondence between the set of all \( f \)-invariant i-v \( L \)-fuzzy prime ideals of \( R \) and the set of all i-v \( L \)-fuzzy prime ideals of \( R' \), where \( R \) and \( R' \) are near-rings and \( f \) is a homomorphism from \( R \) onto \( R' \).

2. Basic Definitions

From now on this paper \( L \) is a complete lattice [3], i.e. there is a partial order \( \leq \) on \( L \) such that, for any \( S \subseteq L \), infimum of \( S \) and supremum of \( S \) exist and these will be denoted by \( \bigwedge_{s \in S} \{s\} \) and \( \bigvee_{s \in S} \{s\} \), respectively. In particular for any elements \( a,b \in L \), in \( f\{a,b\} \) and \( \sup\{a,b\} \) will be denoted by \( a \wedge b \) and \( a \vee b \), respectively. Also, \( L \) is a distributive lattice with a least element 0 and a greatest element 1. If \( a,b \in L \); we write \( a \geq b \) if \( b \leq a \), and \( a > b \) if \( a \geq b \) and \( a \neq b \).

Definition 2.1. Given two elements \( a,b \in L \) with \( a \leq b \), we define the following closed interval set:

\[
[a,b] = \{c \in L | a \leq c \leq b\}.
\]

Suppose \( D(L) \) denotes the family of all closed intervals of \( L \).

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* E-mail: davvaz@yazduni.net
Definition 2.2. Let \( I_1 = [a_i, b_i], \ I_2 = [a_2, b_2] \) and \( I_1 = [a_i, b_i] \) be elements of \( \mathcal{D}(L) \) then we define

\[
I_1 \wedge I_2 = [a_1 \wedge a_2, b_1 \wedge b_2],
\]

\[
I_1 \vee I_2 = [a_1 \vee a_2, b_1 \vee b_2],
\]

\[
\bigwedge_i \{I_i\} = \left[ \bigwedge_i \{a_i\}, \bigwedge_i \{b_i\} \right],
\]

\[
\bigvee_i \{I_i\} = \left[ \bigvee_i \{a_i\}, \bigvee_i \{b_i\} \right].
\]

We call \( I_2 \leq I_1 \) if and only if \( a_2 \leq a_1 \) and \( b_2 \leq b_1 \).

Definition 2.3. Let \( X \) be a non-empty set. An \( L \)-fuzzy subset \( F \) defined on \( X \) is given by

\[
F = \{(x, \mu_F(x)) | x \in X \}, \text{ where } \mu_F : X \rightarrow L.
\]

Definition 2.4. Let \( X \) be a non-empty set. An interval-valued \( L \)-fuzzy subset \( F \) defined on \( X \) is given by

\[
F = \{(x, [\mu^L_k(x), \mu^U_k(x)]) | x \in X \},
\]

where \( \mu^L_k \) and \( \mu^U_k \) are two \( L \)-fuzzy subsets of \( X \) such that \( \mu^L_k(x) \leq \mu^U_k(x) \) for all \( x \in X \).

Suppose \( \hat{\mu}_F(x) = [\mu^L_k(x), \mu^U_k(x)] \). If \( \mu^L_k(x) = \mu^U_k(x) = c \) where \( 0 \leq c \leq 1 \), then we have \( \hat{\mu}_F(x) = [c, c] \) which we also assume, for the sake of convenience, to belong to \( \mathcal{D}(L) \). Thus \( \hat{\mu}_F(x) \in \mathcal{D}(L) \) for all \( x \in X \). Therefore the i-v fuzzy subset \( F \) is given by

\[
F = \{(x, \hat{\mu}_F(x)) | x \in X \}, \text{ where } \hat{\mu}_F : X \rightarrow \mathcal{D}(L).
\]

Definition 2.5. Let \( f \) be a mapping from a set \( X \) into a set \( Y \). Let \( A \) be an i-v \( L \)-fuzzy subset of \( X \) then the image of \( A \), i.e., \( f[A] \) is the i-v fuzzy subset of \( Y \) with the membership function defined by

\[
\hat{\mu}_{f[A]}(y) = \left\{ \begin{array}{ll}
\bigvee_{z \in f^{-1}(y)} \hat{\mu}_A(z) & \text{if } f^{-1}(y) \neq \emptyset \\
[0,0] & \text{otherwise}
\end{array} \right.
\]

for all \( y \in Y \).

Let \( B \) be an i-v \( L \)-fuzzy subset of \( Y \). Then the inverse image of \( B \), i.e., \( f^{-1}[B] \) is the i-v \( L \)-fuzzy subset of \( X \) with the membership function given by

\[
\hat{\mu}_{f^{-1}[B]}(x) = \hat{\mu}_B(f(x)) \quad \text{for all } x \in X.
\]

Definition 2.6. Let \( X \) and \( Y \) be any two non-empty sets and \( f : X \rightarrow Y \) be any function. An i-v \( L \)-fuzzy subset of \( F \) of \( X \) is called \( f \)-invariant if

\[
f(x) = f(y) \Rightarrow \hat{\mu}_F(x) = \hat{\mu}_F(y), \quad \text{where } x, y \in X.
\]

Definition 2.7. A non-empty set \( R \) with two binary operations + and \( \cdot \) is called a near-ring [5,15] if

1) \( (R,+) \) is a group,
2) \( (R, \cdot) \) is a semigroup,
3) \( x \cdot (y+z) = x \cdot y + x \cdot z \) for all \( x, y, z \in R \).

To be more precise, they are left near-rings because the left distributive law is satisfied. We will use the word near-ring to mean left near-ring. We denote \( xy \) instead of \( x \cdot y \). Note that \( x0 = 0 \) and \( x(-y) = -xy \) but in general \( 0x \neq 0 \) for all \( x \in R \) [15, Lemma 1.10]. A near-ring \( R \) is called a zero symmetric if \( 0x = 0 \) for all \( x \in R \).

Definition 2.8. Let \( (R,+,-) \) be a near-ring. An ideal of \( R \) is a subset \( I \) of \( R \) such that

1) \( (I,+,-) \) is a normal subgroup of \( (R,+,-) \),
2) \( RI \subseteq I \),
3) \( (r+i)s - rs \in I \) for all \( i \in I \) and \( r, s \in R \).

Note that if \( I \) satisfies (1) and (2) then it is called a left ideal of \( R \). If \( I \) satisfies (1) and (3) then it is called a right ideal of \( R \). Let \( P \) be an ideal of \( R \). We call \( P \) a prime ideal if for any ideal \( I, J \subseteq R \), \( IJ \subseteq P \) then \( I \subseteq P \) or \( J \subseteq P \).

i-v \( L \)-Fuzzy Ideals in a Near-Ring

In this section we define interval-valued \( L \)-fuzzy subnear-rings and ideals and then we explain some results in this connection.

Definition 3.1. Let \( (R,+,-) \) be a near-ring. An i-v \( L \)-fuzzy subset \( F \) of \( R \) is called an i-v \( L \)-fuzzy subnear-ring, if the following hold:

1) \( \hat{\mu}_F(x) \wedge \hat{\mu}_F(y) \leq \hat{\mu}_F(x-y) \) for all \( x, y \in R \),
2) $\hat{\mu}_F(x) \land \hat{\mu}_F(y) \leq \hat{\mu}_F(x \cdot y)$ for all $x, y \in R$.

Furthermore $F$ is called an i-v $\mathcal{L}$-fuzzy ideal of $R$, if $F$ is an i-v $\mathcal{L}$-fuzzy subnear-ring of $R$ and

3) $\hat{\mu}_F(x) = \hat{\mu}_F(y + x - y)$ for all $x, y \in R$,

4) $\hat{\mu}_F(x) \leq \hat{\mu}_F(xy)$ for all $x, y \in R$,

5) $\hat{\mu}_F(i) \leq \hat{\mu}_F((x + i)y - xy)$ for all $x, y, i \in R$.

Note that $F$ is an i-v $\mathcal{L}$-fuzzy left ideal of $R$ if it satisfies (1), (3) and (4), and $F$ is an i-v $\mathcal{L}$-fuzzy right ideal of $R$ if it satisfies (1), (2), (3) and (5).

Now, we give an example of an i-v $\mathcal{L}$-fuzzy ideal of a near-ring.

**Example 3.2.** Let $R = \{0, a, b, c\}$ be a set with two binary operations as follows:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
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<tr>
<td>a</td>
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<td>b</td>
<td>b</td>
<td>c</td>
<td>a</td>
<td>0</td>
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<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>0</td>
<td>a</td>
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Then $(R, +, \cdot)$ is a near-ring. Define an i-v $\mathcal{L}$-fuzzy subset $F$ by membership function $\hat{\mu}_F : R \to \mathcal{D}(\mathcal{L})$ by $\hat{\mu}_F(b) = \hat{\mu}_F(c) < \hat{\mu}_F(a) < \hat{\mu}_F(0)$. Then $F$ is an i-v $\mathcal{L}$-fuzzy ideal of $R$.

**Lemma 3.3.** For an i-v $\mathcal{L}$-fuzzy ideal $F$ of a near-ring $R$, we have

$\hat{\mu}_F(x) = \hat{\mu}_F(-x) \leq \hat{\mu}_F(0)$ for all $x \in R$.

**Proposition 3.4.** Let $F$ be an i-v $\mathcal{L}$-fuzzy ideal of $R$. If $\hat{\mu}_F(x) = \hat{\mu}_F(0)$ then $\hat{\mu}_F(x) = \hat{\mu}_F(y)$.

**Proof.** Assume that $\hat{\mu}_F(x - y) = \hat{\mu}_F(0)$. Then

$\hat{\mu}_F(x) = \hat{\mu}_F(x - y + y)$

$\geq \hat{\mu}_F(x - y) \land \hat{\mu}_F(y)$

$= \hat{\mu}_F(0) \land \hat{\mu}_F(y)$

$= \hat{\mu}_F(y)$.

Similarly, using $\hat{\mu}_F(y - x) = \hat{\mu}_F(x - y) = \hat{\mu}_F(0)$, we get $\hat{\mu}_F(y) \geq \hat{\mu}_F(x)$.

**Corollary 3.5.** $[\mu^L_F, \mu^R_F]$ is an i-v $\mathcal{L}$-fuzzy ideal of a near-ring $R$ if and only if $\mu^L_F, \mu^R_F$ are $\mathcal{L}$-fuzzy ideals of $R$. Now, we define

$F^L_t = \{x \in X | \mu^L_F(x) \geq t\}$ and

$F^U_s = \{x \in X | \mu^R_F(x) \geq s\}$.

Then $\hat{\mu}_F$ is an i-v $\mathcal{L}$-fuzzy ideal of $R$ if and only if for every $t, s$ where $0 \leq t \leq s \leq 1$, $F^L_t, F^U_s \neq \emptyset$ are ideals of $R$.

**Definition 3.6.** Let $F_1$ and $F_2$ be two i-v $\mathcal{L}$-fuzzy subsets of a near-ring $R$. Then $F_1 \cap F_2$ and $F_1 \cap F_2$ are defined as follows:

$\hat{\mu}_{F_1 \cap F_2} = \hat{\mu}_{F_1}(x) \land \hat{\mu}_{F_2}(x)$,

$\hat{\mu}_{F_1 \cap F_2} = \bigvee_{x \in yz} \{\hat{\mu}_{F_1}(y) \land \hat{\mu}_{F_2}(z)\}$.

$\hat{\mu}_{F_1 \cup F_2} (x) = \begin{cases} 0 & \text{if } x \text{ is not expressible as } x = yz \end{cases}$.

**Lemma 3.7.** Let $R$ be a near-ring, we have

1) If $F_1, F_2$ are two i-v $\mathcal{L}$-fuzzy ideals of $R$ (right or left) then $F_1 \cap F_2$ is an i-v $\mathcal{L}$-fuzzy ideal of $R$ (right or left), respectively;

2) If $R$ is a zero-symmetric and if $F_1$ is an i-v $\mathcal{L}$-fuzzy right ideal and $F_2$ is an i-v $\mathcal{L}$-fuzzy left ideal, then $F_1 \cap F_2 \subseteq F_1 \cap F_2$.

**Proof.** (1) It is an immediate consequence of Corollary 3.5 and Definition 3.6.

(2) We assume $R$ is a zero symmetric near-ring. If $\hat{\mu}_{F_1 \cap F_2}(x) = 0$, there is nothing to prove. Otherwise

$\hat{\mu}_{F_1 \cap F_2} (x) = \bigvee_{x \in yz} \{\hat{\mu}_{F_1}(y) \land \hat{\mu}_{F_2}(z)\}$.

Since $F_1$ is an i-v $\mathcal{L}$-fuzzy left ideal, we have

$\hat{\mu}_{F_1}(z) \leq \hat{\mu}_{F_1}(yz) = \hat{\mu}_{F_1}(x)$,

and since $F_1$ is an i-v $\mathcal{L}$-fuzzy right ideal, we have

$\hat{\mu}_{F_1}(x) = \hat{\mu}_{F_1}(yz) = \hat{\mu}_{F_1}((0 + y)z - 0z) \geq \hat{\mu}_{F_1}(y)$.

Therefore
\[ \hat{\mu}_{F^a|F^b} (x) \leq \hat{\mu}_{F^a} (x) \land \hat{\mu}_{F^b} (x) = \hat{\mu}_{F^a \cap F^b} (x). \]

**Definition 3.8.** Let \( X \) be a non-empty set and \( F \) be an i-v \( \mathcal{L} \)-fuzzy subset of \( X \). Then we define
\[
F_{[t,s]} = \{ x \in X | \hat{\mu}_{F} (x) \geq [t,s] \}.
\]
The set \( F_{[t,s]} \) is called the “level set” of \( F \).

It is easy to see that \( F_{[t,s]} \cap F_{[t',s']} = F_{[\max(t,t'), \min(s,s')]} \).

Now, we obtain the relation between an i-v \( \mathcal{L} \)-fuzzy ideal and level ideals. This relation is expressed in terms of a necessary and sufficient condition.

**Theorem 3.9.** Let \( R \) be a near-ring and \( F \) an i-v \( \mathcal{L} \)-fuzzy subset of \( R \). Then \( F \) is an i-v \( \mathcal{L} \)-fuzzy ideal of \( R \) if and only if for every \( t, s \) where \( 0 \leq t \leq s \leq 1 \), \( F_{[t,s]} \neq \emptyset \) is an ideal of \( R \).

**Proof.** The proof is similar to the proof of Theorem 3.4 of [7], considering the suitable modification with using Definitions 2.4 and 3.1.

**Definition 3.10.** An i-v \( \mathcal{L} \)-fuzzy ideal \( P \) of a near-ring \( R \) is said to be prime if \( P \) is not constant function and for any i-v \( \mathcal{L} \)-fuzzy ideals \( F_1, F_2 \) in \( R \), \( F_1 \cap F_2 \subseteq P \) implies \( F_1 \subseteq P \) or \( F_2 \subseteq P \).

**Proposition 3.11.** Let \( P \) be an i-v \( \mathcal{L} \)-fuzzy prime ideal of a near-ring \( R \). Define
\[
\pi = \{ x \in R | \hat{\mu}_{P} (x) = \hat{\mu}_{P} (0) \},
\]
then \( \pi \) is a prime ideal in \( R \).

**Proof.** The proof is similar to the proof of Theorem 3.7 in [1].

**Proposition 3.12.** Let \( R \) be a near-ring and \( F_1, F_2 \) are i-v \( \mathcal{L} \)-fuzzy prime ideals of \( R \), then \( F_1 \cap F_2 \) is an i-v \( \mathcal{L} \)-fuzzy prime if and only if \( F_1 \subseteq F_2 \) or \( F_2 \subseteq F_1 \).

**Proof.** The proof is straightforward, in view of the fact that \( F_1 \cap F_2 \subseteq F_1 \cap F_2 \).

We have the following corollary which plays an important role in the determination of i-v \( \mathcal{L} \)-fuzzy prime ideals.

**Corollary 3.13.** Let \( R \) be a near-ring. Then every ideal of \( R \) is a level ideal of an i-v \( \mathcal{L} \)-fuzzy ideal of \( R \).

**Proof.** Let \( I \) be any ideal of a near-ring \( R \) and let \( [a_1, a_2] \leq [\beta_1, \beta_2] \neq [0,0] \) be elements in \( \mathcal{D}(\mathcal{L}) \). Then the fuzzy subset \( F \) is defined as follows:
\[
\hat{\mu}_{F} (x) = \begin{cases} 
[\beta_1, \beta_2] & \text{if } x \in I \\
[a_1, a_2] & \text{otherwise.}
\end{cases}
\]

We have \( I = F_{[\beta_1, \beta_2]} \) and by Theorem 3.9, it is enough to prove that \( F \) is an i-v \( \mathcal{L} \)-fuzzy ideal.

An element \( [a_1, a_2] \neq [1,1] \) in \( \mathcal{D}(\mathcal{L}) \) is called “prime” if for any \( [a_1, a_2], [b_1, b_2] \in \mathcal{D}(\mathcal{L}) \), \( [a_1, a_2] \land [b_1, b_2] \leq [a_1, a_2] \) implies either \( [a_1, a_2] \leq [a_1, a_2] \) or \( [b_1, b_2] \leq [a_1, a_2] \).

**Theorem 3.14.** Let \( I \) be a prime ideal of a near-ring \( R \) and let \( [a_1, a_2] \) be a prime element in \( \mathcal{D}(\mathcal{L}) \). Let \( P \) be the fuzzy subset of \( R \) defined by
\[
\hat{\mu}_{P} (x) = \begin{cases} 
[1,1] & \text{if } x \in I \\
[a_1, a_2] & \text{otherwise.}
\end{cases}
\]

Then \( P \) is an i-v \( \mathcal{L} \)-fuzzy prime ideal.

**Proof.** By Corollary 3.13, \( P \) is clearly a non-constant i-v \( \mathcal{L} \)-fuzzy ideal. Let \( F_1 \) and \( F_2 \) be any i-v \( \mathcal{L} \)-fuzzy ideals and let \( F_1 \subseteq P, F_2 \subseteq P \). Then there exist \( x, y \) in \( R \), such that \( \hat{\mu}_{R} (x) \neq \hat{\mu}_{P} (x) \) and \( \hat{\mu}_{F_2} (x) \neq \hat{\mu}_{P} (x) \). This implies that \( \hat{\mu}_{F_1} (x) = \hat{\mu}_{P} (y) = [a_1, a_2] \) and hence \( x \not\in R \) and \( y \not\in R \). Since \( I \) is prime, there exists \( r \in R \) such that \( x r y \not\in I \). Now, we have \( \hat{\mu}_{F_1} (x) \leq [a_1, a_2] \) and \( \hat{\mu}_{F_2} (y) \leq [a_1, a_2] \) (otherwise \( \hat{\mu}_{F_1} (x) \leq [a_1, a_2] \) and since \( [a_1, a_2] \) is prime, \( \hat{\mu}_{F_1} (x) \land \hat{\mu}_{F_2} (y) \leq [a_1, a_2] \) and hence \( (F_1 \cap F_2)(x r y) \leq [a_1, a_2] = \hat{\mu}_{P} (x y) \) so that \( F_1 \cap F_2 \subseteq P \). Hence \( P \) is an i-v \( \mathcal{L} \)-fuzzy prime.

**Lemma 3.15.** Let \( f \) be a mapping from a non-empty set \( X \) into a non-empty set \( Y \), and let \( A, B \) be i-v \( \mathcal{L} \)-fuzzy subsets of \( X, Y \), respectively, such that
\[
\hat{\mu}_A = [\mu_A^L, \mu_A^U] : X \to \mathcal{D}(\mathcal{L}) \quad \text{and} \quad \hat{\mu}_B = [\mu_B^L, \mu_B^U] : Y \to \mathcal{D}(\mathcal{L}).
\]

Then
\[
\hat{\mu}_{f[A]} = [f(\mu_A^L), f(\mu_A^U)] \quad \text{and}
\]
Using Lemma 3.15, the following propositions are obvious.

**Proposition 3.16.** Let \( f \) be a homomorphism from a near ring \( R \) onto a near-ring \( R' \), and \( A \) be any \( f \)-invariant \( i \)-\( v \) \( L \)-fuzzy prime ideal of \( R \). Then \( f(A) \) is an \( i \)-\( v \) \( L \)-fuzzy prime ideal of \( R' \).

**Proposition 3.17.** Let \( f \) be a homomorphism from a near ring \( R \) onto a near-ring \( R' \), and \( B \) be any \( f \)-invariant \( i \)-\( v \) \( L \)-fuzzy prime ideal of \( R' \). Then \( f^{-1}(B) \) is an \( i \)-\( v \) \( L \)-fuzzy prime ideal of \( R \).

**Theorem 3.18.** Let \( f \) be a homomorphism from a near ring \( R \) onto a near-ring \( R' \), then the mapping \( A \to f(A) \) defines a one-to-one correspondence between the set of all \( f \)-invariant \( i \)-\( v \) \( L \)-fuzzy prime ideals of \( R \) and the set of all \( i \)-\( v \) \( L \)-fuzzy prime ideals of \( R' \).

**References**