EXISTENCE OF A STEADY FLOW WITH A BOUNDED VORTEX IN AN UNBOUNDED DOMAIN

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Abstract

We prove the existence of steady 2-dimensional flows, containing a bounded vortex, and approaching a uniform flow at infinity. The data prescribed is the rearrangement class of the vorticity field. The corresponding stream function satisfies a semilinear elliptic partial differential equation. The result is proved by maximizing the kinetic energy over all flows whose vorticity fields are rearrangements of a prescribed function.

Introduction

In this paper we prove the existence of steady 2-dimensional ideal fluid flows occupying Π_+ (the first quadrant) and containing a bounded vortex. Such a flow will be described by a stream function $\psi : \Pi_+ \to \mathbb{R}$. At infinity we will have $\psi \to -\lambda x_1 x_2$ which is the stream function for an irrotational flow with velocity field $-\lambda(x_1, -x_2)$. The vorticity is given by $-\Delta\psi$, where Δ is the Laplacian, and $-\Delta\psi$ vanishes outside a bounded region avoiding the boundary of Π_+. We will show that ψ satisfies the following semilinear elliptic partial differential equation:

$$-\Delta\psi = \phi \circ \psi,$$

almost everywhere in Π_+, where ϕ is an increasing function, unknown *a prior*. In our result the vorticity field $\zeta(= -\Delta\psi)$ is a rearrangement of a prescribed non-negative function ζ_0 having bounded support. The existence theorem is proved by maximizing a functional over the set of rearrangements of ζ_0 vanishing outside bounded sets in Π_+. This variational principle was adapted by Burton [1] from one for vortex rings in 3 dimensions, proposed by Benjamin [2].

Lack of compactness caused by the unboundedness of the domain of interest is the motivation to use the strategy proposed by Benjamin [2].

Keywords: Rearrangements, Vorticity, Irrotational flows, Elliptic partial differential equations, Variational problem

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Notation and Definitions

Henceforth p denotes a real number in $(2,\infty)$ and $p^* := p/(p-1)$. The upper and the right half planes are designated by Π_u and Π_r, respectively, and the first quadrant by Π_+. Generic points of $\mathbb{R}^2$ are denoted by $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$, etc. For $x \in \mathbb{R}^2$ we let $\overline{x}, \underline{x}, \underline{\overline{x}}$ denote the reflections of x about the x_1-axis, x_2-axis and the origin, respectively. For $\xi > 0$ we define

$$\Pi_+(\xi) := \{x \in \mathbb{R}^2 | 0 < x_1 < \xi, 0 < x_2 < \xi\}.$$

The ball centered at x with radius r is denoted $B_r(x)$, when the origin is the center we write B_r.

Here we deal with three different Green's functions, namely, the Green's functions for $-\Delta$ with homogeneous Dirichlet boundary conditions on $\prod_u$, $\prod_r$ and $\prod_+$; it is a standard result that these functions are given as follows

$$G_u(x,y) := \frac{1}{2\pi}\log\frac{|x-\bar{y}|}{|x-y|}, \qquad x,y\in\prod_u, \quad x\neq y,$$

$$G_r(x,y) := \frac{1}{2\pi}\log\frac{|x-\underline{y}|}{|x-y|}, \qquad x,y\in\prod_r, \quad x\neq y,$$

$$G_+(x,y) := \frac{1}{2\pi}\log\frac{|x-\bar{y}||x-\underline{y}|}{|x-y||x-\underline{\bar{y}}|}, \quad x,y\in\prod_+, \quad x\neq y,$$

respectively. For measurable functions ζ on $\mathbb{R}^2$ we define the following integral operators

$$K_u\zeta(x) := \int_{\prod_u} G_u(x,y)\zeta(y)dy,$$

$$K_r\zeta(x) := \int_{\prod_r} G_r(x,y)\zeta(y)dy,$$

$$K_+\zeta(x) := \int_{\prod_+} G_+(x,y)\zeta(y)dy,$$

whenever the integrals exit.

For a measurable set A in $\mathbb{R}^2$, we use $|A|$ to denote the 2-dimensional Lebesgue measure of A, and $\overline{A}$ for the topological closure. The strong support of a measurable function ζ, denoted supp(ζ), is defined as follows

$$\text{supp}(\zeta) := \{x\in \text{dom}(\zeta) | \zeta(x)>0\}.$$

To define the rearrangement classes needed for our variational problem we fix a non-negative, non-trivial function $\zeta_0 \in L^p(\mathbb{R}^2)$) which vanishes outside a bounded set; in addition we assume

$$|\text{supp}(\zeta_0)| = \pi a^2 \tag{1}$$

for some $a>0$. The set $\mathcal{F}$ comprises the functions (which we call *rearrangements* of ζ_0) that vanish outside bounded subsets of $\prod_+$ and that are equimeasurable to ζ_0. A function ζ is said to be equimeasurable to ζ_0 whenever

$$|\{x\in\prod_+ | \zeta(x)\geq\alpha\}| = |\{x\in\prod_+ | \zeta_0(x)\geq\alpha\}|,$$

for every positive α. It is well known that if $\zeta\in\mathcal{F}$ then

$$\|\zeta\|_s = \|\zeta_0\|_s,$$

for $s\in[1,\infty]$. The subset of $\mathcal{F}$ comprising functions vanishing outside $\prod_+(\xi)$ is designated by $\mathcal{F}(\xi)$, where it is assumed that $\xi\geq a\pi^{1/2}$ to ensure $\mathcal{F}(\xi)\neq\emptyset$.

Next we define the *kinetic energy*. For $v\in L^p(\prod_+)$ having bounded support and $\lambda\in\mathbb{R}$, we define

$$\Psi(v) := \frac{1}{2}\int_{\prod_+} vK_+v,$$

$$\Im(v) := \int_{\prod_+} x_1x_2v$$

and the kinetic energy

$$\Psi_\lambda(v) := \Psi(v) - \lambda\Im(v),$$

whenever the integrals exist. Now we are in a position to define the variational problem

$$(P_\lambda): \sup_{\zeta\in\mathcal{F}} \Psi_\lambda(\zeta).$$

The set of solutions of (P_λ) is denoted by Σ_λ. For $\xi > a\pi^{1/2}$ the truncated variational problem $(P_\lambda(\xi))$ is defined by

$$(P_\lambda(\xi)): \sup_{\zeta\in\mathcal{F}(\xi)} \Psi_\lambda(\zeta),$$

and $\Sigma_\lambda(\xi)$ denotes the set of solutions.

Proofs of Some Lemmas

Lemma 1. *Let $\zeta\in L^p(\prod_+)$ vanish outside a bounded set. Then*

(i) $K_+\zeta\in C^1(\mathbb{R}^2)$.

(ii) $|\nabla K_+\zeta(x)|\leq C\|\zeta\|_p$, *for every* $x\in\mathbb{R}^2$, *where C depends on* $|supp(\zeta)|$ *and p.*

(iii) $|K_+\zeta(x)|\leq C\min\{|x_1|,|x_2|\}\|\zeta\|_p$, *for every* $x\in\prod_+$, *where C is the constant in (ii).*

Proof. (i) is an immediate consequence of a result about Newtonian potentials of densities with compact support. Specifically, let

$$N\zeta_e(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\frac{1}{|x-y|}\zeta_e(y)dy$$

denote the Newtonian potential of the zero-extension of ζ, to all of $\mathbb{R}^2$. Since $p>2$ and ζ has compact support we can apply Lemmas A.7 an A.9 in [3] to deduce that $N\zeta_e\in C^1(\mathbb{R}^2)$ and

$$\nabla N\zeta_e(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\nabla_x \log\frac{1}{|x-y|}\zeta_e(y)dy,\quad \forall x\in\mathbb{R}^2. \quad (2)$$

Clearly we have

$$K_+\zeta_e(x)=N\zeta_e(x)+N\zeta_e(\underline{x})-N\zeta_e(\overline{x})-N\zeta_e(\underline{x}).$$

Hence, $K_+\zeta\in C^1(\mathbb{R}^2)$. For (ii) it obviously suffices to show

$$|\nabla N\zeta_e(x)|\le C\|\zeta\|_p,\quad \forall x\in\mathbb{R}^2. \quad (3)$$

where C is a constant depending on $|\text{supp}(\zeta)|$ and p. To do this, we use (2) to deduce

$$|\nabla N\zeta_e(x)|\le\int_{\mathbb{R}^2}\frac{1}{|x-y|}|\zeta(y)|dy,\quad \forall x\in\mathbb{R}^2.$$

Now let us fix $x\in\mathbb{R}^2$ and denote the Schwartz-rearrangement of $|\zeta|$, about x, by ζ^*. Therefore, by a standard inequality, see for example [4], we obtain

$$|\nabla N\zeta_e(x)|\le\frac{1}{2\pi}\int_{B_l(x)}\frac{1}{|x-y|}\zeta^*(y)dy,$$

where

$$l:=(|\text{supp}(\zeta)|/\pi)^{1/2}.$$

Hence, by an application of Hölder's inequality we derive

$$|\nabla N\zeta_e(x)|\le\frac{1}{2\pi}\left(\int_{B_l(x)}\frac{1}{|x-y|^{p^*}}dy\right)^{1/p^*}\|\zeta\|_p. \quad (4)$$

Elementary calculations yield

$$\int_{B_l(x)}\frac{1}{|x-y|^{p^*}}dy\le C,$$

where C depends only on l and p. So we derive (3). Now to derive (iii) we fix $x\in\prod_+$. Since $K_+\zeta\in C^1(\mathbb{R}^2)$ and vanishes on the boundary of $\prod_+$, we can apply the Mean Value Theorem to obtain

$$|K_+\zeta(x)|=|K_+\zeta(x)-K_+\zeta(x_1,0)|\le x_2|\nabla K_+\zeta(\hat{x})|,$$

where $\hat{x}$ is a point on the segment joining x to $(x_1,0)$. Whence from (ii) we deduce that $|K_+\zeta(x)|\le C\,x_2\|\zeta\|_p$. Similarly, one can show $|K_+\zeta(x)|\le C\,x_1\|\zeta\|_p$, so we derive (iii) as desired.◊

Lemma 2. *Let U be an open and bounded subset of $\prod_+$. Then for every $q\ge 1$, $K_+:L^p(U)\to L^q(U)$ is a compact linear operator, in the sense that if $\{\zeta_n\}$ is a sequence of functions, bounded in $L^p(\prod_+)$ and vanishing outside U, then the restrictions to U of the $K\zeta_n$'s have a subsequence converging in the q-norm.*

Proof. The well-defindness of $K_+:L^p(U)\to L^q(U)$ follows from Lemma 1(iii). The linearity of K_+ follows from the definition. To show compactness of K_+ it suffices to show that $K_+:L^p(U)\to W^{1,2}(U)$ is bounded, since then by an application of the Sobolev Embedding Theorem we derive the desired result. Let us now fix $\zeta\in L^p(\prod_+)$ that vanishes outside U. Then by applying Lemma 1(ii), (iii) we infer $\|K_+\zeta\|_2\le C\|\zeta\|_p$ and $\|\nabla K_+\zeta\|_2\le C\|\zeta\|_p$, hence

$$\|K_+\zeta\|_{W^{1,2}(U)}\le C\|\zeta\|_p,$$

here C stands for different constants. So we are done.◊

The next lemma is an immediate consequence of Lemma 7 in [1].

Lemma 3. *Let $\zeta\in L^p(\prod_+)$ vanishes outside a bounded set. Then*

$$\nabla K_+\zeta(x)=O(|x^{-2}|),\quad K_+\zeta(x)=O(|x^{-1}|),\quad |x|\to\infty.$$

Lemma 4. *Let q and U be as in Lemma 2. Then $K_+:L^p(U)\to L^q(U)$ is strictly positive, that is, for every non-trivial function $\zeta\in L^p(\prod_+)$ vanishing outside U,*

$$\int_{\prod_+}\zeta K_+\zeta>0.$$

Proof. Let us fix $\zeta\in L^p(\prod_+)$ vanishing outside U. Then, from Lemma 3(i) in [1], we have

$$-\Delta K_u\zeta=\zeta,\quad in\ \mathcal{D}'(\textstyle\prod_u),$$

that is, in the sense of distributions. Hence, we also have

$$-\Delta K_u\zeta=\zeta,\quad in\ \mathcal{D}'(\textstyle\prod_+),$$

since $K_+\zeta(x)=K_u\zeta(x)-K_u\zeta(\underline{x})$ for all $x\in\mathbb{R}^2$. Now by Agmon's regularity theory [5] we deduce that $K_+\zeta\in W^{2,p}_{loc}(\mathbb{R}^2)$. In particular, $K_+\zeta\in W^{2,p}_{loc}(\prod_+)$. Therefore, in fact we have

$$-\Delta K_u \zeta = \zeta ,$$

almost everywhere in $\prod_+$. Next we let $\Omega(R) := B_R \cap \prod_+$; since the boundary of $\Omega(R)$ is Lipschitz we can apply the weak Divergence Theorem, see for example [6], to obtain

$$-\int_{\Omega(R)} \zeta K_+\zeta + \int_{\Omega(R)} |\nabla K_+\zeta|^2 = \int_{\partial\Omega(R)} \gamma(K_+\zeta)\gamma\left(\partial_{\vec{n}} K_+\zeta\right) d\sigma,$$

where γ stands for the trace operator on $\partial\Omega(R)$ and $\vec{n}$ denotes the unit outward normal vector to $\partial\Omega(R)$. Since $K_+\zeta \in C^1\left(\overline{\prod_+}\right)$ we have

$$\int_{\partial\Omega(R)} \gamma(K_+\zeta)\gamma\left(\partial_{\vec{n}} K_+\zeta\right) d\sigma = \int_{\partial\Omega(R)} (K_+\zeta)\left(\partial_{\vec{n}} K_+\zeta\right) d\sigma .$$

Therefore

$$-\int_{\Omega(R)} \zeta K_+\zeta + \int_{\Omega(R)} |\nabla K_+\zeta|^2 = \int_{\partial\Omega(R)} (K_+\zeta)\left(\partial_{\vec{n}} K_+\zeta\right) d\sigma .$$

Now from Lemma 3 we infer

$$\lim_{R\to\infty} \int_{\partial\Omega(R)} (K_+\zeta)\left(\partial_{\vec{n}} K_+\zeta\right) d\sigma = 0 .$$

Moreover, since $\int_{\prod_+} \zeta K_+\zeta$ is finite and $|\nabla K_+\zeta|$ is bounded in $\mathbb{R}^2$ we may apply the Lebesgue Dominated Convergence Theorem to conclude

$$\lim_{R\to\infty} \int_{\Omega(R)} \zeta K_+\zeta = \int_{\prod_+} \zeta K_+\zeta,$$
$$\lim_{R\to\infty} \int_{\Omega(R)} |\nabla K_+\zeta|^2 = \int_{\prod_+} |\nabla K_+\zeta|^2.$$

Therefore, we derive $\int_{\prod_+} \zeta K_+\zeta = \int_{\prod_+} |\nabla K_+\zeta|^2$ and we are done.◊

The following lemma is a result from Burton's theory [7].

Lemma 5. *Suppose* $\Phi : L^p(\prod_+(\xi)) \to \mathbb{R}$ *is a weakly sequentially continuous, strictly convex functional. Then* $\Phi(\zeta)$ *attains a maximum relative to* $\zeta \in \mathcal{F}(\xi)$, *for* $\xi \ge a\pi^{1/2}$. *Moreover, if* $\hat{\zeta}$ *is a maximizer and* $\psi \in \partial\Phi(\hat{\zeta})$, *the subdifferential of* Φ *at* $\hat{\zeta}$, *then*

$$\hat{\zeta} = \phi \circ \psi ,$$

almost everywhere in $\prod_+(\xi)$, *for some increasing function* ϕ.

Lemma 6. *For every* $\lambda > 0$ *and* $\xi \ge a\pi^{1/2}$, *the problem* $P_\lambda(\xi)$ *is solvable. Moreover, if* $\zeta \in \sum_\lambda(\xi)$, *then*

$$\zeta = \phi \circ (K_+\zeta - \lambda x_1 x_2), \tag{5}$$

almost everywhere in $\prod_+$, *for some increasing function* ϕ.

Proof. Let us being by noting that $K_+ : L^p(\prod_+(\xi)) \to L^{p^*}(\prod_+(\xi))$ is a symmetric operator, that is,

$$\int_{\prod_+} v\, K_+ w = \int_{\prod_+} w\, K_+ v , \quad \forall\, v, w \in L^p(\prod_+),$$

which readily follows from the symmetry of G_+. Since K_+ is compact, strictly positive and symmetric it follows that Ψ_λ, defined on the set of functions in $L^p(\prod_+)$ vanishing outside $\prod_+(\xi)$, is strictly convex and weakly sequentially continuous. Now by applying Lemma 5 we deduce that $P_\lambda(\xi)$ is solvable. Next we show that if $\zeta \in \sum_\lambda(\xi)$, then $K_+\zeta - \lambda x_1 x_2 \in \partial\Psi_\lambda(\zeta)$. For this purpose we consider $\bar{\zeta} \in L^p(\prod_+)$ which vanishes outside $\prod_+(\xi)$, then we need to show that

$$\Psi_\lambda(\bar{\zeta}) \ge \Psi_\lambda(\zeta) + \int_{\prod_+} (\bar{\zeta} - \zeta)(K_+\zeta - \lambda x_1 x_2),$$

or equivalently

$$\int_{\prod_+} (\bar{\zeta} - \zeta)\, K_+(\bar{\zeta} - \zeta) \ge 0 ,$$

but this is true since K_+ is strictly positive. Therefore, again by Lemma 5, existence of an increasing function ϕ is ensured so that (5) holds.◊

Results and Discussion

In this section we present our main result (see the theorem below). We begin with some technical lemmas.

Lemma 7. *Let* $\lambda > 0$. *Then there exists* $R(\lambda) > 0$ *such that*

$$K_+\zeta(x) - \lambda x_1 x_2 \le 0, \quad |x| \ge R(\lambda), \quad \zeta \in \mathcal{F} .$$

Proof. Let us fix $x \in \prod_+$ and $\zeta \in \mathcal{F}$; we assume $\min\{x_1, x_2\} \ge \alpha$ for some $\alpha > 0$ to be determined later. According to Lemma 1(ii) there exists $M > 0$, independent of ζ, such that $|\nabla K_+\zeta(x)| \le M$. Therefore, by Lemma 1(iii) we have $|K_+\zeta(x)| \le M\, min\{x_1, x_2\}$.

Hence

$$K_+\zeta(x)-\lambda x_1x_2 \le min\{x_1,x_2\}(M-\lambda\alpha).$$

Thus if we assume $\alpha \ge M/\lambda$, then $K_+\zeta(x)-\lambda x_1x_2 \le 0$. Hence we can take $R(\lambda)=M/\lambda$.◊

Lemma 8. *Suppose $\zeta \in L^p(\mathbb{R}^2)$ is a non-negative function which is spherically decreasing and vanishes outside B_a. Then there exists a positive constant k such that*

$$\int_{\Pi_+}\zeta_t K_+\zeta_t \ge k\log t,$$

for all sufficiently large t, where $\zeta_t(x):=\zeta(x_1-t,x_2-t)$.

Proof. Clearly we can assume $t \ge (1+\sqrt{2})a$. Now we observe that there exists $\beta>0$ and $0<b<a$ such that for all x with $|x|\le b$ we have $\zeta(x)\ge\beta$. Let $B_b(t)$ denote the ball centered at (t,t) with radius b and consider $x\in B_a(t)$ and $y \in B_b(t)$. Hence if we set

$$\gamma_1 := |x-\overline{y}|, \gamma_2 := |x-\underline{y}|, \gamma_3 := |x-y|, \gamma_4 := |x-\underline{\overline{y}}|,$$

then it is clear that

$$\gamma_1 \ge 2t-2a, \gamma_2 \ge 2t-2a, \gamma_3 \le 2a, \gamma_4 \le 2\sqrt{2}t+2a.$$

Therefore,

$$K_+\zeta_t(x) \ge \frac{\beta}{2\pi}\int_{B_b(t)}\log\frac{(2t-2a)^2}{2a\left(2\sqrt{2}t+2a\right)}dy = \frac{\beta b^2}{2}\log\frac{(t-a)^2}{a\left(\sqrt{2}t+a\right)}$$

Hence

$$\int_{\Pi_+}\zeta_t K_+\zeta_t \ge \frac{\pi\beta^2 b^4}{2}\log\frac{(t-a)^2}{a(\sqrt{2}t+a)},$$

and we are done.◊

An immediate consequence of Lemma 8 is the following

Corollary. *We have*

$$\lim_{\xi\to\infty}\sup_{\zeta\in\mathcal{F}(\xi)}\Psi(\zeta)=+\infty.$$

Lemma 9. *There exists $\lambda_0>0$ and $\xi_0>a\pi^{1/2}$ such that, if $0<\lambda\le\lambda_0$, $\xi\ge\xi_0$ and $\zeta_{\lambda,\xi}$ is a maximizer of $\Psi_\lambda(\zeta)$ relative to $\zeta\in\mathcal{F}(\xi)$ then*

$$\left|\left\{x\in\Pi_+(\xi)\middle|K_+\zeta_{\lambda,\xi}(x)-\lambda x_1x_2>0\right\}\right| \ge \pi a^2.$$

Proof. Let us fix $\alpha>0$, $\varepsilon>0$. Then according to the Corollary there exists $\xi_0>a\pi^{1/2}$ such that if $\xi\ge\xi_0$ then $\sup_{\zeta\in\mathcal{F}(\xi)}\Psi(\zeta)\ge\alpha+\varepsilon$. In particular, $\sup_{\zeta\in\mathcal{F}(\xi_0)}\Psi(\zeta)\ge\alpha+\varepsilon$. Since Ψ is a real-valued functional on $L^p(\Pi_+(\xi_0))$ which is weakly sequentially continuous and strictly convex we can apply Lemma 5 to ensure existence of $\hat{\zeta}\in\mathcal{F}(\zeta_0)$ *such that* $\Psi(\hat{\zeta})=\sup_{\zeta\in\mathcal{F}(\zeta_0)}\Psi(\zeta)$, whence

$$\Psi(\hat{\zeta})\ge\alpha+\varepsilon. \tag{6}$$

Now choose $\lambda_0>0$ such that $\lambda_0\Im(\hat{\zeta})<\varepsilon$. Since $\Psi_\lambda(\hat{\zeta}):=\Psi(\hat{\zeta})-\lambda\Im(\hat{\zeta})$ we can use (6) to obtain

$$\Psi_\lambda(\hat{\zeta})\ge\alpha, \quad 0<\lambda\le\lambda_0.$$

This shows that

$$\sup_{\zeta\in\mathcal{F}(\xi)}\Psi_\lambda(\zeta)\ge\alpha, \quad 0<\lambda\le\lambda_0, \quad \xi\ge\xi_0. \tag{7}$$

Next we set $\alpha=3aC\|\zeta_0\|_p\,\|\zeta_0\|_1$, where C is the constant in Lemma 1(iii). Hence from (7) we have

$$\sup_{\zeta\in\mathcal{F}(\xi)}\Psi_\lambda(\zeta)\ge 3aC\|\zeta_0\|_p\,\|\zeta_0\|_1, \tag{8}$$

for all $0<\lambda\le\lambda_0$ and $\xi\ge\xi_0$. Now we fix $0<\lambda\le\lambda_0$, $\xi\ge\xi_0$ and let $\zeta_{\lambda,\xi}$ denote a maximizer of $\Psi_\lambda(\zeta)$ relative to $\zeta\in\mathcal{F}(\xi)$. Then we have

$$\Psi_\lambda(\zeta_{\lambda,\xi})\le\|\zeta_0\|_1\sup_{\Pi_+(\xi)}\left(\frac{1}{2}K_+\zeta_{\lambda,\xi}(x)-\lambda x_1x_2\right).$$

We also have $\Psi_\lambda(\zeta_{\lambda,\xi})\ge 3aC\|\zeta_0\|_p\,\|\zeta_0\|_1$, from (8), hence

$$\sup_{\Pi_+(\xi)}\left(\frac{1}{2}K_+\zeta_{\lambda,\xi}(x)-\lambda x_1x_2\right)>3aC\|\zeta_0\|_p. \tag{9}$$

Since $\frac{1}{2}K_+\zeta_{\lambda,\xi}(x)-\lambda x_1x_2\in\left(\overline{\Pi_+(\xi)}\right)$, it attains its maximum at $(x_1^0,x_2^0)\in\overline{\Pi_+(\xi)}$, say. Whence, by Lemma 1(iii)

$$\sup_{\Pi_+(\xi)}\left(\frac{1}{2}K_+\zeta_{\lambda,\xi}(x)-\lambda x_1x_2\right)\le\frac{1}{2}K_+\zeta_{\lambda,\xi}\left(x_1^0,x_2^0\right)$$
$$\le\frac{C}{2}\min\{x_1^0,x_2^0\}\|\zeta_0\|_p .$$

Therefore, from (9) we infer $\min\{x_1^0,x_2^0\}\ge 6a>2a$. Now we define the set

$$S:=\{x\in\Pi_+ | x_1<x_1^0, x_2<x_2^0\}\cap B_{2a}\left(x_1^0,x_2^0\right),$$

where $B_{2a}\left(x_1^0,x_2^0\right)$ denotes the ball centered at $\left(x_1^0,x_2^0\right)$ with radius 2*a*; clearly $S\subset\overline{\Pi}_+(\xi)$. Consider $x\in S$, then

$$K_+\zeta_{\lambda,\xi}(x)-\lambda x_1x_2\ge 1/2K_+\zeta_{\lambda,\xi}(x)-\lambda x_1^0,x_2^0 . \tag{10}$$

On the other hand, by an application of the Mean Value Theorem and Lemma 1(ii),

$$\left|K_+\zeta_{\lambda,\xi}(x)-K_+\zeta_{\lambda,\xi}\left(x_1^0,x_2^0\right)\right|\le\left|\nabla K_+\zeta_{\lambda,\xi}(\hat{x})\right|\left|x-\left(x_1^0,x_2^0\right)\right|$$
$$\le 2aC\|\zeta_0\|_p ,$$

where $\hat{x}$ is a point on the segment joining *x* to $\left(x_1^0,x_2^0\right)$, whence

$$K_+\zeta_{\lambda,\xi}(x)\ge K_+\zeta_{\lambda,\xi}\left(x_1^0,x_2^0\right)-2aC\|\zeta_0\|_p .$$

This, in turn, implies

$$K_+\zeta_{\lambda,\xi}(x)\ge K_+\zeta_{\lambda,\xi}\left(x_1^0,x_2^0\right)-2aC\|\zeta_0\|_p . \tag{11}$$

Thus from (8), (10) and (11) we infer

$$K_+\zeta_{\lambda,\xi}(x)-\lambda x_1x_2\ge\frac{1}{2}K_+\zeta_{\lambda,\xi}\left(x_1^0,x_2^0\right)-aC\|\zeta_0\|_p-\lambda x_1^0x_2^0$$
$$\ge 3aC\|\zeta_0\|_p-aC\|\zeta_0\|_p=2aC\|\zeta_0\|_p.$$

Therefore, $S\subseteq\operatorname{supp}\left(K_+\zeta_{\lambda,\xi}(x)-\lambda x_1x_2\right)$. Hence

$$\left|\operatorname{supp}\left(K_+\zeta_{\lambda,\xi}(x)-\lambda x_1x_2\right)\right|\ge|S|=\pi a^2,$$

as desired.◊

Theorem. *There exists* $\lambda_0>0$ *such that* $\sum_\lambda\ne\emptyset$, *for* $\lambda\in(0,\lambda_0)$. *Moreover, if* $\zeta\in\sum_\lambda$ *and* $\psi:=K_+\zeta$, *then* ψ *satisfies the following elliptic partial differential equation*

$$-\Delta\psi=\phi\circ(\psi-\lambda x_1x_2),\quad a.e.\ in\ \Pi_+, \tag{12}$$

for some increasing function ϕ, *unknown a priori.*

Proof. Let ξ_0 and λ_0 be as in Lemma 9. If we fix $\lambda\in(0,\lambda_0)$, then by Lemma 7 there exists $R(\lambda)>0$ such that

$$K_+\zeta_{\lambda,\xi}(x)-\lambda x_1x_2\le 0,$$
$$x\in\Pi_+\setminus\Pi_+(R(\lambda)),\quad\zeta\in\mathcal{F}. \tag{13}$$

Next we define $\xi^*:=\max\{\xi_0,R(\lambda)\}$; then according to Lemma 6, $\Psi_\lambda(\zeta)$ has a maximizer relative to $\zeta\in\mathcal{F}(\xi^*)$, say ζ_{λ,ξ^*}. For simplicity we write $\hat{\zeta}:=\zeta_{\lambda,\xi^*}$. We claim that $\hat{\zeta}\in\sum_\lambda$. To prove this, suppose $l\ge\xi^*$ and consider $\overline{\zeta}\in\sum_\lambda(l)$. We will first show that

$$\operatorname{supp}\left(\overline{\zeta}\right)\subseteq\Pi_+\left(\xi^*\right), \tag{14}$$

modulo a set of measure zero. By Lemma 6 there sxists an increasing function $\overline{\phi}$ such that

$$\overline{\zeta}=\overline{\phi}\circ\left(K_+\overline{\zeta}-\lambda x_1x_2\right), \tag{15}$$

almost everywhere in $\Pi_+(l)$. Next we observe that since $\overline{\phi}$ is increasing, $\left(\overline{\phi}\right)^{-1}(0,\infty]$, the pre-image of $(0,\infty]$ under $\overline{\phi}$, is an interval, say *I*, of the form (c,∞) or $[c,\infty)$, by assuming that $\overline{\phi}$ takes on the value $+\infty$ on the interval $\left(\left\|K_+\overline{\zeta}-\lambda x_1x_2\right\|_{\infty,\overline{\Pi_+(l)}},\infty\right)$. This, along with (15), implies

$$\operatorname{supp}\left(\overline{\zeta}\right)=\left(K_+\overline{\zeta}-\lambda x_1x_2\right)^{-1}(I),$$

modulo a set of measure zero in $\Pi_+(l)$. Hence $\left|\left(K_+\overline{\zeta}-\lambda x_1x_2\right)^{-1}(I)\right|=\pi a^2$. On the other hand, from Lemma 9, we have $\left|\left(K_+\overline{\zeta}-\lambda x_1x_2\right)^{-1}(0,\infty)\right|\ge\pi a^2$. Whence $c\ge 0$, and this implies $\operatorname{supp}\left(\overline{\zeta}\right)\subseteq\operatorname{supp}(K_+\overline{\zeta}-\lambda x_1x_2)$ modulo a set of measure zero in $\Pi_+(l)$. Finally, according to (13), we also have $\operatorname{supp}(K_+\overline{\zeta}-\lambda x_1x_2)\subseteq\Pi_+\left(\xi^*\right)$, hence we derive (14). Now from (14) we infer $\Psi_\lambda(\hat{\zeta})\ge\Psi_\lambda(\overline{\zeta})$ and this, in turn, implies that $\overline{\zeta}\in\sum_\lambda(l)$. Since $l\ge\xi^*$ is arbitrary we deduce that $\hat{\zeta}\in\sum_\lambda$.

To derive (12) we use Lemma 6 once again to ensure existence of an increasing function $\hat{\phi}$ such that

$$\hat{\zeta}=\hat{\phi}\circ\left(K_+\hat{\zeta}-\lambda x_1x_2\right),$$

almost everywhere in $\prod_+(\xi^*)$. We obtain (12) by a modification process, that is, define

$$\phi(s) := \begin{cases} \hat{\phi}(s), & s \in dom(\hat{\phi}), \quad s > 0, \\ 0, & s \leq 0. \end{cases}$$

therefore, clearly, we have

$$\hat{\zeta} = \phi \circ \left(K_+\hat{\zeta} - \lambda x_1 x_2 \right),$$

almost everywhere in $\prod_+$. As required.◊

Let us conclude with the following

Remark. A close inspection of the proofs of the Theorem and Lemma 9 confirms that if $\zeta \in \Sigma_\lambda$, then

$$\text{supp}(\zeta) \subseteq \left\{ x \in \prod_+ \middle| K_+\zeta(x) - \lambda x_1 x_2 \geq 2aC\|\zeta_0\|_p \right\},$$

modulo a set of measure zero. Hence for almost every $x \in \text{supp}(\zeta)$ we have

$$2aC\|\zeta_0\|_p \leq K_+\zeta(x) - \lambda x_1 x_2 \leq K_+\zeta(x) \\ \leq C \min\{x_1, x_2\} \|\zeta_0\|_p,$$

where in the last inequality we have used Lemma 1(iii). Therefore, for almost every $x \in \text{supp}(\zeta)$

$$\min\{x_1, x_2\} \geq 2a \, .$$

This shows that the vortex core essentially avoids the boundary of $\prod_+$.

Similar problems have been considered in Emamizadeh [8,9].

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References

1. Burton, G. R. Steady symmetric vortex pairs and rearrangements. *Proceedings of the Royal Society of Edinburgh,* **108A**, 369-290, (1988).
2. Benjamin, T. B. The alliance of practical and analytical insights into the nonlinear problems of fluid mechanics, Applications of methods of functional analysis to problems in mechanics. *Lecture Notes in Mathematics 503,* Springer-Verlag, 8-29, (1979).
3. Fraenkel, L. E. An Introduction to Maximum Principles and Symmetry in Elliptic Problems: Cambridge University Press. (In the press).
4. Hardy, G. H. Littlewood, Polya, G. Inequalities. Cambridge University Press, (1952).
5. Agmon, S. The L^p approaching to the Dirichlet problem. *Ann. Scuola Norm. Pisa Cl. Sci.,* **13**(3), 405-448, (1959).
6. Grisvard, P. Singularities in Boundary Value Problems. Masson, Springer-Verlag, (1992).
7. Burton, G. R. Rearrangements of functions, maximization of convex functionals and vortex rings. *Math. Ann.,* **276**, 225-253, (1987).
8. Burton, G. R. Emamizadeh, B. A constrained variational problem for steady vortices in a shear flow. *Commun. Partial Diff. Eq.,* **24**(7/8), 1341-65, (1999).
9. Emamizadeh, B. Steady vortex in a uniform shear flow of an ideal fluid. *Proc. Roy. Soc. Edin.,* **130A**, 801-812, (2000).