

UNIVERSAL COMPACTIFICATIONS OF TRANSFORMATION SEMIGROUPS*

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Abstract

We extend the notion of semigroup compactification to the class of transformation semigroups, and determine the compactifications which are universal with respect to some topological properties.

1. Introduction and Preliminaries

Throughout the paper, (S, X) denotes a transformation semigroup, i.e., a semigroup S endowed with a topology, a topological space X , and an action $(s, x) \rightarrow sx: S \times X \rightarrow X$ such that $s(tx) = (st)x$ for all $s, t \in S$ and $x \in X$. If T is a subsemigroup of S and Y is a T -invariant subspace of X (i.e., $TY \subseteq Y$), we then call (T, Y) a *sub-transformation semigroup* of (S, X) . (T, Y) is said to be dense in (S, X) if both T is dense in S and Y is dense in X .

For $s, t \in S, x \in X$, we define the translation maps $\lambda_s: S \rightarrow S$, $\rho_s: S \rightarrow S$, $\dot{\lambda}_s: X \rightarrow X$ and $\dot{\rho}_x: S \rightarrow X$ by $\lambda_s(t) = st = \rho_t(s)$ and $\dot{\lambda}_s(x) = sx = \dot{\rho}_x(s)$. We say (S, X) is *left topological* if λ_s and $\dot{\lambda}_s$ are continuous for all $s \in S$, *right topological* if ρ_s and $\dot{\rho}_x$ are continuous for all $s \in S, x \in X$, *semitopological* if (S, X) is both left and right topological, *topological* if the multiplication in S and the action of S on X are continuous. (S, X) is said to be compact (resp. Hausdorff) if so are both S and X .

Keywords: Semitopological transformation semigroup; Admissible pair of function spaces; Universal compactification

For example, if $S := X^X$ is the semigroup (under composition) of selfmaps of a topological space X (with the topology of pointwise convergence on X), then under the natural action $(s, x) \rightarrow s(x)$, (S, X) is a right topological transformation semigroup, and (T, Y) is a semitopological sub-transformation semigroup of (S, X) , where $T = \{s \in S: \dot{\lambda}_s \text{ is continuous}\}$ and Y is any T -invariant subspace of X .

The general approach to the theory of semigroup compactification is based on the Gelfand-Naimark theory of commutative C^* -algebras. Consequently, compactifications of a semitopological transformation semigroup (S, X) appear as pairs of the spectra of certain C^* -algebras of functions on S and X respectively, such that the first spectrum as a semigroup acts on the second one. Almost (and weakly almost), periodic compactifications of (S, X) are studied in Junghenn [5], and Pourabdollah [7].

The literature on harmonic analysis on transformation semigroups is in fact very limited. The main reason, we believe, lies in the scarcity of concrete non-trivial examples of transformation semigroups. As far as we know, Berglund and Hofmann [1] is the first major work dealing with harmonic analysis on transformation semigroups. Milnes [6] provides some strong results which, in particular, show the nature of some well-

* AMS Subject Classification: 43A60, 54H15

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known transformation groups and transformation semigroups in connection with weak almost periodicity. The pioneering work of Ellis [3] provides a substantial introduction to algebraic and topological properties of transformation groups.

The reader may notice that we have touched the subject of transformation semigroup compactification in its most general form. Naturally, one would expect to obtain much sharper results under some additional conditions; such as, local compactness of S and/or X , or in the transformation group setting. We hope to deal with these special cases in a forthcoming paper.

For notation and terminology we shall follow Berglund *et al.* [2], as far as possible.

By a *homomorphism* from (S, X) into a transformation semigroup (T, Y) we mean a pair (ϕ, ψ) where $\phi: S \rightarrow T$ is a semigroup homomorphism and $\psi: X \rightarrow Y$ is a map with the property $\psi(sx) = \phi(s)\psi(x)$ for each $s \in S$ and $x \in X$. In this case, $(\phi(S), \psi(X))$ is a sub-transformation semigroup of (T, Y) . We say (ϕ, ψ) is one-to-one (resp. onto) if so are both ϕ and ψ . A transformation semigroup homomorphism that is one-to-one and onto is called an *isomorphism*. (ϕ, ψ) is said to be continuous if so are ϕ and ψ .

2. Transformation Semigroups of Means

Let $C(X)$ denote the C^* -algebra of all bounded continuous complex-valued functions on X . Recall that for a linear subspace \mathcal{F} of $C(X)$, the space $M(\mathcal{F}) = \{\mu \in \mathcal{F}^* : \|\mu\| = \mu(1) = 1\}$ of means on \mathcal{F} is convex and weak*-compact in \mathcal{F}^* , the evaluation map $\delta: X \rightarrow M(\mathcal{F})$, defined for all $f \in \mathcal{F}$ by $\delta(x)(f) = f(x)$ is weak*-continuous, the convex hull $\text{co}(\delta(X))$ of $\delta(X)$ is dense in $M(\mathcal{F})$, the convex circled hull $\text{cco}(\delta(X))$ of $\delta(X)$ is dense in the closed unit ball of \mathcal{F}^* , and \mathcal{F}^* is the closed linear span of $\delta(X)$. If \mathcal{F} is also an algebra, the space $MM(\mathcal{F})$ of multiplicative means on \mathcal{F} (=the spectrum of \mathcal{F}) is compact, and $\delta(X)$ is dense in $MM(\mathcal{F})$. We denote $MM(C(X))$ by βX .

For $s \in S$ and $x \in X$, we consider the translation operators $L_s = \lambda_s^*$ and $R_s = \rho_s^*$ on $C(S)$; $\dot{L}_s = (\dot{\lambda}_s)^*$ and $\dot{R}_x = (\dot{\rho}_x)^*$ on $C(X)$ (here $*$ denotes the dual mapping; for example $\dot{\lambda}_s^*(f) = f \circ \dot{\lambda}_s$, for each $f \in C(X)$, and so forth). Trivially $\dot{L}_{st} = \dot{L}_t \dot{L}_s$, $\dot{R}_{sx} = R_s \dot{R}_x$, $\dot{L}_s \dot{R}_x = \dot{R}_x \dot{L}_s$.

A subset \mathcal{F} of $C(S)$ (resp. \mathcal{H} of $C(X)$) is called *translation invariant* if $L_s \mathcal{F} \cup R_s \mathcal{F} \subseteq \mathcal{F}$ (resp. $\dot{L}_s \mathcal{H} \subseteq$

\mathcal{H}) for all $s \in S$.

In what follows, let $\mathcal{F} \subseteq C(S)$ and $\mathcal{H} \subseteq C(X)$ be translation invariant C^* -subalgebras, containing the constant functions, and let $\dot{R}_x \mathcal{H} \subseteq \mathcal{F}$ for all $x \in X$. For $\mu \in \mathcal{F}^*$ and $\eta \in \mathcal{H}^*$ we define the maps T_μ and U_μ on \mathcal{F} , \dot{T}_η and \dot{U}_μ on \mathcal{H} by $T_\mu f(s) = \mu(L_s f)$, $U_\mu f(s) = \mu(R_s f)$, $(\dot{T}_\eta h)(s) = \eta(\dot{L}_s h)$, and $(\dot{U}_\mu h)(x) = \mu(\dot{R}_x h)$, where $s \in S$, $x \in X$, $f \in \mathcal{F}$, $h \in \mathcal{H}$. These are bounded linear operators, and $\dot{T}_{\delta(x)} = \dot{R}_x$, $\dot{T}_\eta \dot{L}_s = L_s \dot{T}_\eta$, $\dot{T}_\eta R_s = \dot{T}_{\eta \circ R_s}$, $\dot{U}_{\varepsilon(s)} = \dot{L}_s$, $U_\mu \dot{R}_x = \dot{R}_x U_\mu$, $\dot{U}_\mu \dot{L}_s = \dot{U}_{\mu \circ L_s}$, where $\varepsilon: S \rightarrow M(\mathcal{F})$ and $\delta: X \rightarrow M(\mathcal{H})$ denote the evaluation maps (see [2]).

We call the pair $(\mathcal{F}, \mathcal{H})$ left (resp. right) *m-admissible* if the inclusions $T_\mu \mathcal{F} \subseteq \mathcal{F}$ and $\dot{T}_\eta \mathcal{H} \subseteq \mathcal{F}$ (resp. $U_\mu \mathcal{F} \subseteq \mathcal{F}$ and $\dot{U}_\mu \mathcal{H} \subseteq \mathcal{H}$) hold for all multiplicative means μ on \mathcal{F} and η on \mathcal{H} . $(\mathcal{F}, \mathcal{H})$ is called *m-admissible* if it is both left and right m-admissible.

Let $(\mathcal{F}, \mathcal{H})$ be a left (resp. right) m-admissible pair of C^* -algebras for (S, X) . For each $\mu, \eta \in \mathcal{F}^*$, $\eta \in \mathcal{H}^*$, we define $\mu \eta := \mu \circ T_\eta$ (resp. $\mu * \eta := \eta \circ U_\mu$) on \mathcal{F} , and $\mu \cdot \eta = \mu \circ \dot{T}_\eta$ (resp. $\mu * \eta = \eta \circ \dot{U}_\mu$) on \mathcal{H} , and obtain a multiplication $(\mu, \nu) \rightarrow \mu \nu$ (resp. $\mu * \nu$) in \mathcal{F}^* , and an action $(\mu, \eta) \rightarrow \mu \cdot \eta$ (resp. $\mu * \eta$) of the semigroup \mathcal{F}^* on \mathcal{H}^* .

The following theorem describes the basic structure of transformation semigroup compactifications. The proof is essentially the same as that of the semigroup case (see [2; 2.2.11(ii)]).

Theorem 2.1. *Let (S, X) be right (resp. left) topological. If $(\mathcal{F}, \mathcal{H})$ is a left (resp. right) m-admissible pair of C^* -algebras for (S, X) then with respect to the weak* topology, the multiplication $(\mu, \nu) \rightarrow \mu \nu$ (resp. $(\mu, \nu) \rightarrow \mu * \nu$), and the action $(\mu, \eta) \rightarrow \mu \cdot \eta$ (resp. $(\mu, \eta) \rightarrow \mu * \eta$), $(MM(\mathcal{F}), MM(\mathcal{H}))$ is a compact right (resp. left) topological transformation semigroup, $(\varepsilon(S), \delta(X))$ is a dense semitopological sub-transformation semigroup of $(MM(\mathcal{F}), MM(\mathcal{H}))$, and $(\varepsilon, \delta): (S, X) \rightarrow (MM(\mathcal{F}), MM(\mathcal{H}))$ is a continuous homomorphism.*

3. Transformation Semigroup Compactifications

We define a right (resp. left) topological compactification of a semitopological (S, X) as a pair $((\phi, \psi), (T, Y))$, where (T, Y) is a compact Hausdorff right (resp.

left) topological transformation semigroup, and $(\phi, \psi): (S, X) \rightarrow (T, Y)$ is a continuous homomorphism such that $(\phi(S), \psi(X))$ is a dense semitopological sub-transformation semigroup of (T, Y) .

Let $((\phi, \psi), (T, Y))$ and $((\phi', \psi'), (T', Y'))$ be right (resp. left) topological compactifications of (S, X) . If $\pi: T \rightarrow T'$ and $\gamma: Y \rightarrow Y'$ are continuous functions such that $\pi \circ \phi = \phi'$ and $\gamma \circ \psi = \psi'$ (it follows easily that (π, γ) is a continuous homomorphism of (T, Y) onto (T', Y')), then (π, γ) is called a *homomorphism* of $((\phi, \psi), (T, Y))$ onto $((\phi', \psi'), (T', Y'))$. If such a homomorphism exists then it is unique, and we say that $((\phi, \psi), (T, Y))$ is an *extension* of $((\phi', \psi'), (T', Y'))$. If (π, γ) is also one-to-one then it is an *isomorphism* of compactifications.

Let P be a property of compactifications of (S, X) . A P -compactification of (S, X) is a compactification of (S, X) having the property P . A P -compactification of (S, X) that is an extension of every P -compactification of (S, X) is called a *universal P -compactification* of (S, X) .

By the inclusion $(\mathcal{F}_1, \mathcal{H}_1) \subseteq (\mathcal{F}_2, \mathcal{H}_2)$ we mean $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{H}_1 \subseteq \mathcal{H}_2$. In particular $(\mathcal{F}_1, \mathcal{H}_1) = (\mathcal{F}_2, \mathcal{H}_2)$ means $\mathcal{F}_1 = \mathcal{F}_2$ and $\mathcal{H}_1 = \mathcal{H}_2$.

Theorem 3.1. *Let (S, X) be semitopological.*

(i) *There is a one-to-one correspondence between right topological compactifications of (S, X) and left m -admissible pairs of C^* -algebras for (S, X) .*

(ii) *If $(\mathcal{F}_i, \mathcal{H}_i)$ is a left m -admissible pairs of C^* -algebras for (S, X) , and $((\phi_i, \psi_i), (T_i, Y_i))$ its corresponding right topological compactification of (S, X) , $i = 1, 2$, then $((\phi_1, \psi_1), (T_1, Y_1))$ is an extension of $((\phi_2, \psi_2), (T_2, Y_2))$ if and only if $(\mathcal{F}_1, \mathcal{H}_1) \supseteq (\mathcal{F}_2, \mathcal{H}_2)$. Hence $((\phi_1, \psi_1), (T_1, Y_1)) = ((\phi_2, \psi_2), (T_2, Y_2))$ if and only if $(\mathcal{F}_1, \mathcal{H}_1) = (\mathcal{F}_2, \mathcal{H}_2)$.*

Similar conclusions hold for dual versions of (i) and (ii).

Proof. (i) Let $((\phi, \psi), (T, Y))$ be a right topological compactification of (S, X) . Since ψ^* is an isometric $*$ -isomorphism, $\mathcal{H} := \psi^*C(Y)$ is a C^* -subalgebra of $C(X)$, and the identity $\dot{L}_s \circ \psi^* = \psi^* \circ \dot{L}_{\phi(s)}$ implies that \mathcal{H} is translation invariant. Let $\eta \in MM(\mathcal{H})$ and $\{x_\alpha\}$ be a net X such that $\delta(x_\alpha) \rightarrow \eta$ in weak* topology. By the compactness of Y , we may assume that $y := \lim_\alpha \psi(x_\alpha)$ exists in Y . Now the identity $\dot{R}_x \circ \psi^* = \psi^* \circ \dot{R}_{\psi(x)}$ together with the fact that $\dot{T}_{\delta(x_\alpha)} = \dot{R}_{x_\alpha}$ for

each α , imply that $\dot{T}_\eta \psi^* = \lim_\alpha \dot{T}_{\delta(x_\alpha)} \psi^* = \lim_\alpha \dot{R}_{x_\alpha} \psi^* = \lim_\alpha \psi^* \circ \dot{R}_{\psi(x_\alpha)} = \psi^* \circ \dot{R}_y$. Since $\dot{R}_y C(Y) \subseteq C(T)$ we have $\dot{T}_\eta \mathcal{H} = \dot{T}_\eta \psi^* C(Y) = \psi^* \circ \dot{R}_y C(Y) \subseteq \psi^* C(T) = \mathcal{F}$. On the other hand, an argument similar to [2; 3.1.7] shows that \mathcal{F} is left m -admissible in the usual (semigroup) case. Therefore, we see that $(\mathcal{F}, \mathcal{H})$ is a left m -admissible pair of C^* -algebras for (S, X) . Conversely, if $(\mathcal{F}, \mathcal{H})$ is a left m -admissible pair of C^* -algebras for (S, X) , then by Theorem 2.1 $((\phi, \psi), (T, Y)) := ((\mathcal{E}, \delta), (MM(\mathcal{F}), MM(\mathcal{H})))$ is a right topological compactification of (S, X) such that $(\phi^* C(T), \psi^* C(Y)) = (\mathcal{F}, \mathcal{H})$. It is unique up to isomorphism, and is denoted by $((\mathcal{E}, \delta), (S^\mathcal{F}, X^\mathcal{H}))$.

(ii) Let (π_1, π_2) from $((\phi_1, \psi_1), (T_1, Y_1))$ onto $((\phi_2, \psi_2), (T_2, Y_2))$ be a homomorphism of right topological compactifications, then $(\mathcal{F}_2, \mathcal{H}_2) = (\phi_2^* C(T_2))$, $(\mathcal{H}_2, \mathcal{Y}_2) = (\psi_2^* C(Y_2)) = (\phi_1^* (\pi_1^* C(T_2)), \psi_1^* (\pi_2^* C(Y_2))) \subseteq (\phi_1^* C(T_1), \psi_1^* C(Y_1)) = (\mathcal{F}_1, \mathcal{H}_1)$.

Conversely, if $\mathcal{F}_2 \subseteq \mathcal{F}_1$ and $\mathcal{H}_2 \subseteq \mathcal{H}_1$, then $V_1 := (\phi_1^*)^{-1} \circ \phi_2^*$ and $V_2 := (\psi_1^*)^{-1} \circ \psi_2^*$ are C^* -algebra isomorphisms of $C(T_2)$ into $C(T_1)$ and $C(Y_2)$ into $C(Y_1)$, respectively. It is a well-known fact that these C^* -algebra homomorphisms are induced by continuous mappings $\pi_1: T_1 \rightarrow T_2$ and $\pi_2: Y_1 \rightarrow Y_2$ such that $\pi_1^* = V_1$ and $\pi_2^* = V_2$, (see [2; 2.1.20]). Then $\phi_1^* \circ \pi_1^* = \phi_2^*$ and $\psi_1^* \circ \pi_2^* = \psi_2^*$, which implies $\pi_1 \circ \phi_1 = \phi_2$ and $\pi_2 \circ \psi_1 = \psi_2$. Therefore, $((\phi_1, \psi_1), (T_1, Y_1))$ is an extension of $((\phi_2, \psi_2), (T_2, Y_2))$. \square

Remark 3.2. For a locally convex topological vector space X , $\mathcal{AF}(X)$ is the Banach subspace of affine functions in $C(X)$. (S, X) is called *affine* if S and X are convex subsets of locally convex topological vector spaces such that $\lambda_s, \dot{\lambda}_s, \rho_s$ and $\dot{\rho}_s$ are affine maps. An affine homomorphism is a homomorphism (ϕ, ψ) of affine transformation semigroups such that ϕ and ψ are affine maps. We call the pair $(\mathcal{F}, \mathcal{H}) \subseteq (C(S), C(X))$ of translation invariant conjugate-closed Banach spaces, left (resp. right) admissible if the inclusions $T_\mu \mathcal{F} \subseteq \mathcal{F}$ and $\dot{T}_\eta \mathcal{H} \subseteq \mathcal{F}$ (resp. $U_\mu \mathcal{F} \subseteq \mathcal{F}$ and $\dot{U}_\mu \mathcal{H} \subseteq \mathcal{H}$) hold for all means μ on \mathcal{F} and η on \mathcal{H} . In this case, $(\mathcal{F}^*, \mathcal{H}^*)$ is a right (resp. left) topological affine transformation semigroup (each endowed with its weak* topology), $(M(\mathcal{F}), M(\mathcal{H}))$ is a compact right (resp. left) topological affine sub-transformation semigroup of $(\mathcal{F}^*, \mathcal{H}^*)$, and $(co(\mathcal{E}(S)), co(\mathcal{E}(X)))$ is a dense semitopological affine sub-transformation semigroup of

$(M(\mathcal{F}), M(\mathcal{H}))$, and $(\varepsilon, \delta): (S, X) \rightarrow (M(\mathcal{F}), M(\mathcal{H}))$ is a continuous homomorphism.

A right (resp. left) topological affine compactification of (S, X) is a pair $((\phi, \psi), (T, Y))$, where (T, Y) is a compact Hausdorff right (resp. left) topological affine transformation semigroup and $(\phi, \psi): (S, X) \rightarrow (T, Y)$ is a continuous homomorphism such that $(co(\phi(S)), co(\psi(X)))$ is a dense semitopological affine sub-transformation semigroup of (T, Y) .

Homomorphisms, isomorphisms, and extensions of right (resp. left) topological affine compactifications are defined similar to non-affine case, with the additional requirement that these are affine mappings.

Similar to the non-affine case, it can be shown that if $((\phi, \psi), (T, Y))$ is a right (resp. left) topological affine compactification of (S, X) , then the pair $(\phi^* \mathcal{AF}(T), \psi^* \mathcal{AF}(Y))$ is a left (resp. right) admissible pair of Banach spaces for (S, X) . Conversely, if $(\mathcal{F}, \mathcal{H})$ is a left (resp. right) admissible pair of Banach spaces for (S, X) , then there exists a unique up to isomorphism right (resp. left) topological affine compactification $((\phi, \psi), (T, Y)) := ((\varepsilon, \delta), (M(\mathcal{F}), M(\mathcal{H})))$ of (S, X) such that $(\phi^* \mathcal{AF}(T), \psi^* \mathcal{AF}(Y)) = (\mathcal{F}, \mathcal{H})$, denoted by $((\varepsilon, \delta), (aS^{\mathcal{F}}, aX^{\mathcal{H}}))$. A universal P -affine compactification of (S, X) is a P -affine compactification that is an extension of every P -affine compactification of (S, X) .

4. Function Spaces and Universal Compactifications

For a semitopological (S, X) we define,

$\mathcal{AP}(X) = \{f \in C(X) : \dot{L}_s f \text{ is norm relatively compact in } C(X)\}$,

$\mathcal{WAP}(X) = \{f \in C(X) : \dot{L}_s f \text{ is weakly relatively compact in } C(X)\}$,

$\mathcal{LC}(X) = \{f \in C(X) \mid s \rightarrow \dot{L}_s f : S \rightarrow C(X) \text{ is norm continuous}\}$,

$\mathcal{RC}(X) = \{f \in C(X) \mid x \rightarrow \dot{R}_x f : X \rightarrow C(S) \text{ is norm continuous}\}$,

$\mathcal{WLC}(X) = \{f \in C(X) \mid s \rightarrow \dot{L}_s f : S \rightarrow C(X) \text{ is } \sigma(C(X), C(X)^*)\text{-continuous}\}$,

$\mathcal{WRC}(X) = \{f \in C(X) \mid x \rightarrow \dot{R}_x f : X \rightarrow C(S) \text{ is } \sigma(C(S), C(S)^*)\text{-continuous}\}$,

$\mathcal{LMC}(X) = \{f \in C(X) \mid s \rightarrow \dot{L}_s f : S \rightarrow C(X) \text{ is } \sigma(C(X), \beta X)\text{-continuous}\}$,

$\mathcal{RMC}(X) = \{f \in C(X) \mid x \rightarrow \dot{R}_x f : X \rightarrow C(S) \text{ is } \sigma(C(S), \beta S)\text{-continuous}\}$.

βS)-continuous}.

Clearly $\mathcal{AP}(X) \subseteq \mathcal{WAP}(X)$, $\mathcal{LC}(X) \subseteq \mathcal{WLC}(X) \subseteq \mathcal{LMC}(X)$ and $\mathcal{RC}(X) \subseteq \mathcal{WRC}(X) \subseteq \mathcal{RMC}(X)$. The inclusions $\mathcal{AP}(X) \subseteq \mathcal{LC}(X) \cap \mathcal{RC}(X)$ and $\mathcal{WAP}(X) \subseteq \mathcal{WLC}(X) \cap \mathcal{WRC}(X)$ follow from part (i) of the following proposition.

Proposition 4.1. (i) For each $f \in \mathcal{WAP}(X)$ (resp. $\mathcal{AP}(X)$), the maps $s \rightarrow \dot{L}_s f : S \rightarrow \mathcal{WAP}(X)$ (resp. $\mathcal{AP}(X)$) and $x \rightarrow \dot{R}_x f : X \rightarrow \mathcal{WAP}(S)$ (resp. $\mathcal{AP}(S)$) are weakly (resp. norm)-continuous.

(ii) If (S, X) is a compact semitopological (resp. topological) transformation semigroup, then for each $f \in C(X)$, $\dot{L}_s f \subseteq C(X)$ and $\dot{R}_x f \subseteq C(S)$ are weakly (resp. norm)-compact, and so $\mathcal{WAP}(X)$ (resp. $\mathcal{AP}(X) = C(X)$).

Proof. (i) We consider the \mathcal{WAP} -case only. Since the map $s \rightarrow \dot{L}_s f : S \rightarrow C(X)$ is obviously pointwise-continuous and $\dot{L}_s f$ is weakly relatively compact, the weak topology and the pointwise topology coincide on $\dot{L}_s f$; hence, $s \rightarrow \dot{L}_s f$ must be norm continuous. The proof for the right translation case is similar (use Theorem 4.2(ii)).

(ii) Note that the maps $s \rightarrow \dot{L}_s f$ and $x \rightarrow \dot{R}_x f$ are weakly (resp. norm) continuous by [2; A.9] (resp. [2; B.3]).

The following gives some characterizations of the desired function spaces, and has a standard proof similar to the semigroup case (see [2; chapter 4]).

Proposition 4.2. Let (S, X) be semitopological and $f \in C(X)$.

(i) The following statements are equivalent.

(a) $f \in \mathcal{AP}(X)$.

(b) $\dot{R}_x f$ is norm relatively compact in $C(S)$.

(c) For each $\varepsilon > 0$, there is a finite $K \subseteq S$ such that

$$\min\{\|\dot{L}_s f - \dot{L}_t f\| : t \in K\} < \varepsilon \quad (s \in S).$$

(d) For each $\varepsilon > 0$, there is a finite $F \subseteq X$ such that

$$\min\{\|\dot{R}_x f - \dot{R}_y f\| : y \in F\} < \varepsilon \quad (x \in X).$$

(ii) The following statements are equivalent.

(a) $f \in \mathcal{WAP}(X)$.

(b) $\dot{R}_x f$ is weakly relatively compact.

(c) $\dot{L}_s f$ is $\sigma(C(X), \beta X)$ relatively compact.

(d) $\dot{R}_x f$ is $\sigma(C(S), \beta S)$ relatively compact.

- (e) $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} f(s_m x_n) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} f(s_m x_n)$ when $\{s_m\} \subseteq S$ and $\{x_m\} \subseteq X$ are sequences such that all the limits exist.
 - (iii) The following statements are equivalent.
 - (a) $f \in \mathcal{LC}(X)$ (resp. $\mathcal{RC}(X)$).
 - (b) $\dot{R}_X f$ (resp. $\dot{L}_S f$) is equicontinuous on S (resp. X).
 - (c) $\text{co}\dot{R}_X f$ (resp. $\text{co}\dot{L}_S f$) is equicontinuous on S (resp. X).
 - (iv) The following statements are equivalent.
 - (a) $f \in \mathcal{LMC}(X)$ (resp. $\mathcal{WLC}(X)$).
 - (b) $\dot{R}_X f$ (resp. $\text{co}\dot{R}_X f$) is pointwise-relatively compact in $C(S)$.
 - (c) $\dot{T}_\eta f \in C(S)$ for all $\eta \in \beta X$ (resp. $M(C(X))$).
 - (v) The following statements are equivalent.
 - (a) $f \in \mathcal{RMC}(X)$ (resp. $\mathcal{WRMC}(X)$).
 - (b) $\dot{L}_S f$ (resp. $\text{co}\dot{L}_S f$) is pointwise relatively compact in $C(X)$.
 - (c) $\dot{U}_\mu f \in C(X)$ for all $\mu \in \beta S$ (resp. $M(C(S))$).
- The following theorem is proved in [4].

Theorem 4.3. Let (S, X) be semitopological. Let $\mathcal{F} \subseteq C(S)$ and $\mathcal{H} \subseteq C(X)$ be translation invariant conjugate-closed Banach subspaces, containing the constant functions, and let $\dot{R}_x \mathcal{H} \subseteq \mathcal{F}$ for all $x \in X$. Let B_1 and \dot{B}_1 denote the closed unit balls of \mathcal{F}^* and \mathcal{H}^* , respectively. Then for any $h \in \mathcal{H}$ the following hold.

- (i) $h \in \mathcal{WAP}(X)$, if and only if, for each $\mu \in \mathcal{F}^*$ and $\eta \in \mathcal{H}^*$, $\dot{T}_\eta h \in \mathcal{F}$, $\dot{U}_\mu h \in \mathcal{H}$, $\mu(\dot{T}_\eta h) = \eta(\dot{U}_\mu h)$
- (ii) $h \in \mathcal{WAP}(X)$ is in $\mathcal{AP}(X)$, if and only if, the complex-valued function $(\mu, \eta) \rightarrow \mu(\dot{T}_\eta h)$ is weak* continuous on $B_1 \times \dot{B}_1$.

Theorem 4.4. Let (S, X) be a semitopological transformation semigroup, then

- (i) $(\mathcal{LMC}(S), \mathcal{LMC}(X))$ (resp. $(\mathcal{RMC}(S), \mathcal{RMC}(X))$) is the largest left (resp. right) m -admissible pair of C^* -algebras,
- (ii) $(\mathcal{WLC}(S), \mathcal{WLC}(X))$ (resp. $(\mathcal{WRC}(S), \mathcal{WRC}(X))$) is the largest left (resp. right) admissible pair of Banach spaces,
- (iii) $(\mathcal{LC}(S), \mathcal{LC}(X))$ (resp. $(\mathcal{RC}(S), \mathcal{RC}(X))$) is a left (resp. right) admissible pair of C^* -algebras,
- (iv) $(\mathcal{WAP}(S), \mathcal{WAP}(X))$ is the largest admissible pair of C^* -algebras for which the equalities $\mu\nu = \mu^* \nu$

and $\mu.\eta = \mu^* \eta$ hold for all $\mu, \nu \in \mathcal{F}^*$, $\eta \in \mathcal{H}^*$,

(v) $(\mathcal{AP}(S), \mathcal{AP}(X))$ is the largest admissible pair of C^* -algebras, in $(\mathcal{WAP}(S), \mathcal{WAP}(X))$, for which the maps $(\mu, \nu) \rightarrow \mu\nu: B \times B \rightarrow \mathcal{F}^*$ and $(\mu, \eta) \rightarrow \mu.\eta: B \times \dot{B} \rightarrow \mathcal{H}^*$ are weak*-continuous for all pairs of norm-bounded sets $(B, \dot{B}) \subseteq (\mathcal{F}^*, \mathcal{H}^*)$.

Proof. By Proposition 4.2(iv) and (v), we have

$$\begin{aligned} \mathcal{WLC}(X) &= \cap \{ \dot{T}_\eta^{-1}(C(S)) : \eta \in M(C(X)) \}, \\ \mathcal{LMC}(X) &= \cap \{ \dot{T}_\eta^{-1}(C(S)) : \eta \in \beta X \}, \\ \mathcal{WRMC}(X) &= \cap \{ \dot{U}_\mu^{-1}(C(X)) : \mu \in M(C(S)) \}, \\ \mathcal{RMC}(X) &= \cap \{ \dot{U}_\mu^{-1}(C(X)) : \mu \in \beta S \}. \end{aligned}$$

Since \dot{T}_η and \dot{U}_μ are bounded linear operators, it follows that $\mathcal{LMC}(X)$, $\mathcal{RMC}(X)$, $\mathcal{WLC}(X)$ and $\mathcal{WRMC}(X)$ are closed linear subspaces of $C(X)$. Clearly, these function spaces contain the constant functions and it is easy to see that they are conjugate-closed and translation invariant. Also, $\mathcal{LMC}(X)$ and $\mathcal{RMC}(X)$ are algebras, since \dot{T}_η and \dot{U}_μ are multiplicative for $\mu \in \beta S$ and $\eta \in \beta X$. Hence, $\mathcal{LMC}(X)$ and $\mathcal{RMC}(X)$ are C^* -subalgebras of $C(X)$.

Now, for $\mu \in \beta S$ and $\eta \in \beta X$ we define a functional $\mu.\eta$ on $\mathcal{LMC}(X)$ by $\mu.\eta(f) = \mu(\dot{T}_\eta f)$. Then $\mu.\eta$ is well-defined, and $\mu.\eta \in MM(\mathcal{LMC}(X))$. Hence, $\mu.\eta$ extends to a member of βX . Denoting this extension also by $\mu.\eta$, we have

$$T_\mu(\dot{T}_\eta f) = \dot{T}_{\mu.\eta} f \quad (f \in \mathcal{LMC}(X), \mu \in \beta S, \eta \in \beta X).$$

Thus, $\dot{T}_\eta(\mathcal{LMC}(X)) \subseteq \mathcal{LMC}(S)$ for all $\eta \in \beta X$. This together with [2; 4.5.2], imply that $(\mathcal{LMC}(S), \mathcal{LMC}(X))$ is left m -admissible. A similar argument shows that $(\mathcal{WLC}(S), \mathcal{WLC}(X))$ is left admissible. (iii) has a routine proof. (iv) and (v) follows essentially from Theorem 4.3 (for more details, see [4]).□

The proof of the following proposition is routine.

Proposition 4.5. Let $(\phi, \psi): (S, X) \rightarrow (T, Y)$ be a continuous homomorphism of semitopological transformation semigroups. Then for $\mathcal{H} = \mathcal{AP}, \mathcal{WAP}, \mathcal{LC}, \mathcal{RC}, \mathcal{WLC}, \mathcal{WRMC}, \mathcal{LMC}$, or \mathcal{RMC} ,

$$\psi^*(\mathcal{H}(Y)) \subseteq \mathcal{H}(X)$$

where $\psi^*: C(Y) \rightarrow C(X)$ denotes the dual map. In

particular if (S, X) is a sub-transformation semigroup of (T, Y) then $\mathcal{H}(Y)|_X \subseteq \mathcal{H}(X)$.

We say a compactification or affine compactification $((\phi, \psi), (T, Y))$ of (S, X) has the joint continuity property if the actions of S on T and on Y , i.e., the maps $(s, t) \rightarrow \phi(s)t : S \times T \rightarrow T, (s, y) \rightarrow \phi(s)y : S \times Y \rightarrow Y$ are (jointly) continuous.

Theorem 4.6. For a semitopological (S, X) , the following statements hold.

(i) $((\varepsilon, \delta), (S^{\mathcal{WAP}}, X^{\mathcal{WAP}}))$ (resp. $((\varepsilon, \delta), (S^{\mathcal{AP}}, X^{\mathcal{AP}}))$) is the universal semitopological (resp. topological) compactification of (S, X) .

(ii) $((\varepsilon, \delta), (S^{\mathcal{LMC}}, X^{\mathcal{LMC}}))$ (resp. $((\varepsilon, \delta), (S^{\mathcal{RMC}}, X^{\mathcal{RMC}}))$) is the universal right (resp. left) topological compactification of (S, X) ,

(iii) $((\varepsilon, \delta), (S^{\mathcal{LC}}, X^{\mathcal{LC}}))$ (resp. $((\varepsilon, \delta), (S^{\mathcal{RC}}, X^{\mathcal{RC}}))$) is the right (resp. left) topological compactification of (S, X) that is universal with respect to the joint continuity property,

(iv) $((\varepsilon, \delta), (aS^{\mathcal{WAP}}, aX^{\mathcal{WAP}}))$ (resp. $((\varepsilon, \delta), (aS^{\mathcal{AP}}, aX^{\mathcal{AP}}))$) is the universal semitopological (resp. topological) affine compactification of (S, X) ,

(v) $((\varepsilon, \delta), (aS^{\mathcal{WLC}}, aX^{\mathcal{WLC}}))$ (resp. $((\varepsilon, \delta), (aS^{\mathcal{WRC}}, aX^{\mathcal{WRC}}))$) is the universal right (resp. left) topological affine compactification of (S, X) ,

(vi) $((\varepsilon, \delta), (aS^{\mathcal{LC}}, aX^{\mathcal{LC}}))$ (resp. $((\varepsilon, \delta), (aS^{\mathcal{RC}}, aX^{\mathcal{RC}}))$) is the right (resp. left) topological affine compactification of (S, X) that is universal with respect to the joint continuity property.

Proof. (i) By Theorem 3.1(i) and Theorem 4.4(iv), $((\varepsilon, \delta), (S^{\mathcal{WAP}}, X^{\mathcal{WAP}}))$ is a semitopological compactification of (S, X) . If $((\phi, \psi), (T, Y))$ is another such compactification then, by Proposition 4.1(ii), $(C(T), C(Y)) = (\mathcal{WAP}(T), \mathcal{WAP}(Y))$. Hence, by Proposition 4.5, $(\phi^*(C(T)), \psi^*(C(Y))) \subseteq (\mathcal{WAP}(S), \mathcal{WAP}(X))$. Thus, by Theorem 3.1(ii), $((\varepsilon, \delta), (S^{\mathcal{WAP}}, X^{\mathcal{WAP}}))$ is an extension of $((\phi, \psi), (T, Y))$. The proof for \mathcal{AP} is similar.

(ii) By Theorem 3.1(i) and Theorem 4.4(i), $((\varepsilon, \delta), (S^{\mathcal{LMC}}, X^{\mathcal{LMC}}))$ is a right topological compactification of (S, X) . If $((\phi, \psi), (T, Y))$ is another such compactification then $(C(T), C(Y)) = (\mathcal{LMC}(T), \mathcal{LMC}(Y))$. Hence, by Proposition 4.5, $(\phi^*(C(T)), \psi^*(C(Y))) \subseteq (\mathcal{LMC}(S), \mathcal{LMC}(X))$. Thus, by Theorem 3.1(ii), $((\varepsilon, \delta), (S^{\mathcal{LMC}},$

$X^{\mathcal{LMC}}))$ is an extension of $((\phi, \psi), (T, Y))$. The proof for \mathcal{RMC} is similar.

(iii) To prove that $((\varepsilon, \delta), (S^{\mathcal{LC}}, X^{\mathcal{LC}}))$ has the joint continuity property, by [2; 4.4.4], it suffices to show that the map $(s, \eta) \rightarrow g(\varepsilon(s)\eta) : S \times X^{\mathcal{LC}} \rightarrow C$ is continuous for each $g \in C(X^{\mathcal{LC}})$. Since $f := \delta(g) \in \mathcal{LC}(X)$, for $s, s_0 \in S$ and $\eta, \eta_0 \in X^{\mathcal{LC}}$ we have $g(\varepsilon(s)\eta) = \eta(\dot{L}_s f)$ and

$$|g(\varepsilon(s)\eta) - g(\varepsilon(s_0)\eta_0)| \leq |\eta(\dot{L}_s f - \dot{L}_{s_0} f)| + |(\eta - \eta_0)(\dot{L}_{s_0} f)| \leq \|\dot{L}_s f - \dot{L}_{s_0} f\| + \|(\eta - \eta_0)(\dot{L}_{s_0} f)\|,$$

which shows $(s, \eta) \rightarrow g(\varepsilon(s)\eta)$ is continuous at (s_0, η_0) .

Now if $((\phi, \psi), (T, Y))$ is any right topological compactification of (S, X) with the joint continuity property then, by [2; B.3], for any $h \in C(Y)$ the map $s \rightarrow \dot{L}_{\phi(s)} h : S \rightarrow C(Y)$ is norm-continuous, and since

$$\|\dot{L}_s \psi^*(h) - \dot{L}_t \psi^*(h)\| = \|\dot{L}_{\psi(s)} h - \dot{L}_{\psi(t)} h\|,$$

we see that $\psi^*(h) \in \mathcal{LC}(X)$. Therefore, $\psi^*(C(Y)) \subseteq \mathcal{LC}(X)$, which by Theorem 3.1(ii), $((\varepsilon, \delta), (S^{\mathcal{LC}}, X^{\mathcal{LC}}))$ is an extension of $((\phi, \psi), (T, Y))$. The proof for \mathcal{RC} is similar.

(iv), (v), and (vi) are affine cases of (i), (ii), and (iii), respectively, which have similar proofs. \square

Acknowledgements

We would like to thank Dr. H. R. Ebrahimi-Vishki for his valuable comments.

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