GENERALIZED POSITIVE DEFINITE FUNCTIONS AND COMPLETELY MONOTONE FUNCTIONS ON FOUNDATION SEMIGROUPS*

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Abstract

A general notion of completely monotone functionals on an ordered Banach algebra B into a proper H^*-algebra A with an integral representation for such functionals is given. As an application of this result we have obtained a characterization for the generalized completely continuous monotone functions on weighted foundation semigroups. A generalized version of Bochner's theorem on foundation semigroups is also obtained.

Introduction

In the present, paper we shall introduce the concept of a completely monotone functional on an ordered Banach algebra B into a proper H^*-algebra A and we shall give an integral representation for such functionals with respect to A-valued measures on $\Delta_+(B)$, the space of all positive multiplicative linear functionals on B. As an application of the theory we shall obtain an integral representation for the generalized w-bounded continuous completely monotone A-valued functions with respect to positive A-valued measures on Γ_w^+, the space of all w-bounded continuous nonnegative semicharacters on a foundation semigroup S with a Borel measurable weight function w. We will also give a generalization of our earlier version of Bochner's theorem [4; Theorem 4.2].

Keywords: Locally compact semigroups; Positive definite functions; H^*-algebras; Spectral measures

1. Preliminaries

Recall that (see, [11], [12], [13], [17]) a proper H^*-algebra is a Banach algebra A whose norm is a Hilbert space norm and which has an involution: $x \to x^*$ on A such that $(y, x^*, z) = (xy, z) = (x, zy^*)$ for all $x, y, z \in A$. Let $\tau(A) = \{xy : x, y \in A\}$ be the trace class of A. It is a Banach algebra with respect to a norm $\tau(.)$ which is related to the norm $\|.\|$ of A by $\tau(a^*a) = \|a\|^2$ for all $a \in A$. There is a trace tr defined on $\tau(A)$ such that $tr(ab) = tr(ba) = (a, b^*)$ for all $a, b \in A$, where (., .) denotes the scalar product on A. if $a = b^*b$ for some $b \in A$ then a is called positive and we write $a \geq 0$. It is obvious that $a \geq 0$ if and only if $(ax, x) \geq 0$ for all $x \in A$. A right module H over A is called a Hilbert module if there is a $\tau(A)$-valued function (,) on $H\times H$ with the following properties

1. $(\xi + \eta, \varphi) = (\xi, \varphi) + (\eta, \varphi)$ for all $\xi, \eta, \varphi \in H$.
2. $(\xi, \eta)^* = (\eta, \xi)$ for all $\xi, \eta \in H$.
3. $(\xi, \eta a) = (\xi, \eta)a$ for all $\xi, \eta \in H$ and each $a \in A$.
4. $(\xi, \xi) \geq 0$ for all $\xi \in H$ and $(\xi, \xi) = 0$ if and only

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if $\xi = 0$.

5. $|tr(\xi,\eta)|^2 \le \tau(\xi,\xi)\tau(\eta,\eta)$ for all $\xi,\eta \in H$.

6. H is complete in the norm $\|\xi\| = (\tau(\xi,\xi))^{1/2}$.

The function (,) is called a generalized scalar product. There is a linear structure on H such that H is an ordinary Hilbert space with respect to the scalar product $\langle \xi,\eta\rangle = tr(\eta,\xi)$. An A-linear operator on H is an additive linear mapping $T: H \to H$ such that $T(\xi a) = (T\xi)a$ for all $\xi \in H, a \in A$; T is bounded in the in the sense that $\|T\xi\| \le M\|\xi\|$ for some $M \ge 0$ and every $\xi \in H$. For each bounded A-linear operator T its adjoint T^* is A-linear and has the property that $(T\xi,\eta) = (\xi, T^*\eta)$ for all $\xi,\eta \in H$.

By a real ordered Banach algebra we shall mean a real Banach algebra $\mathcal{M}_r$ with a closed partial order $\ge$ satisfying the following:

(i) $x \ge y \Rightarrow x + z \ge y + z$, for all $z \in \mathcal{M}_r$.

(ii) $x \ge 0, y \ge 0 \Rightarrow xy \ge 0$.

(iii) $x \ge 0 \Rightarrow ax \ge 0$, for all nonnegative real numbers α.

Note that an order "$\ge$" on an ordered Banach algebra is called closed if for every two sequences (x_n) and (y_n) in B from $x_n \xrightarrow{n} x$ and $y_n \xrightarrow{n} y$ and $x_n \ge y_n$ $(n \in \mathbb{N})$ it follows that $x \ge y$. A complex Banach algebra B of the form $\mathcal{M}_r \oplus i\mathcal{M}_r$, where $\mathcal{M}_r$ is a real ordered Banach algebra, is called an ordered Banach algebra. On an ordered Banach algebra B, we put $P(B) = \{b \in B : b \ge 0\}$ and $P_1(B) = \{b \in P(B) : \|b\| = 1\}$. A linear functional f on B is called positive if $f(b) \ge 0$ for all $b \in P(B)$. In the case where B is commutative, we shall denote by $\Delta(B)$ the space of all bounded multiplicative linear functionals on B and by $\Delta_+(B)$ the space of all positive functionals in $\Delta(B)$.

Definition 1.1. Let B be a commutative ordered Banach algebra. For every $n \in \mathbb{Z}_+$ (the set of nonnegative integers) we define the operator Δ_n on B^* (the dual of B) by

$$\Delta_0 f(b) = f(b)$$

$$\Delta_1 f(b;b_1) = \Delta_0 f(b) - \Delta_0 f(bb_1) = f(b) - f(bb_1)$$

and for every $n \ge 2$

$$\Delta_n f(b;b_1,\dots,b_n) = \Delta_{n-1} f(b;b_1,\dots,b_{n-1}) - \Delta_{n-1} f(bb_n;b_1,\dots,b_{n-1})$$

$(f \in B^*, b, b_1,\dots,b_n \in B; n = 1,2,\dots)$. A linear functional $f \in B^*$ is called *completely monotone* if

$$\Delta_n f(b;b_1,\dots,b_n) \ge 0$$

for all $n \in \mathbb{Z}_+$ and $b, b_1,\dots,b_n \in P_1(B)$.

An operator-valued transformation $U: B \to \mathcal{L}(H)$ (the space of all bounded linear operators on a Hilbert space H) is called *completely monotone* if for every $\xi \in H$ the mapping $\varphi_\xi : b \mapsto \langle U_b\xi,\xi\rangle (b \in B)$ defines a completely monotone functional on B.

We now recall some definitions concerning topological semigroups.

Throughout this paper S will denote a locally compact, Hausdorff topological semigroup.

Definition 1.2. On a commutative topological semigroup S with $C_b(S)$ (the space of bounded continuous complex-valued functions on S) inductive identity, for each $n \in \mathbb{Z}_+$ we define the operator Δ_n on by

$$\Delta_0 f(x) = f(x),$$

$$\Delta_1 f(x;h_1) = \Delta_0 f(x) - \Delta_0 f(xh_1) = f(x) - f(xh_1)$$

and for every $n \ge 2$

$$\Delta_n f(x;h_1,\dots,h_n) = \Delta_{n-1} f(x;h_1,\dots,h_{n-1}) - \Delta_{n-1} f(xh_n;\dots,h_{n-1}),$$

$(f \in C_b(S), x, h_1,\dots,h_n \in S, n = 1,2,\dots)$. A function $f \in C_b(S)$ is called *completely monotone* if $\Delta_n f \ge 0$ $(n \in \mathbb{Z}_+)$ (cf. [5; p. 43]).

Definition 1.3. An operator-valued transformation $T: S \to \mathcal{L}(H)$ is called *completely monotone* if for every $\xi \in H$ the mapping

$$x \mapsto \langle T_x\xi,\xi\rangle (x \in S)$$

is completely monotone on S.

Definition 1.4. Let B be an ordered commutative Banach algebra and H be a Hilbert module over a proper H^*-algebra A. A linear mapping $f: B \to A$ is called a *completely monotone A-functional* if for every $n \in \mathbb{Z}_+$ $\Delta_n f(b;b_1,\dots,b_n) \ge 0$ for every $(n+1)$-positive elements $b, b_1,\dots,b_n$ of B where,

$$\Delta_0 f(b) = f(b)$$

$$\Delta_1 f(b;b_1) = \Delta_0 f(b) - \Delta_0 f(bb_1) = f(b) - f(bb_1)$$

and for every $n \ge 2$

$$\Delta_n f(b;b_1,\ldots,b_n) = \Delta_{n-1} f(b;b_1,\ldots,b_{n-1}) - \Delta_{n-1} f(bb_n;b_1,\ldots,b_{n-1}).$$

Definition 1.5. Let S be a commutative topological semigroup with an identity. A mapping $f: S \to A$ is called *completely monotone* if $\Delta_n f(x;h_1,\ldots,h_n) \geq 0$ for all nonnegative integers n and all $x,h_1,\ldots,h_n \in S$ where

$$\Delta_0 f(x) = f(x)$$

$$\Delta_1 f(x;h_1) = \Delta_0 f(x) - \Delta_0 f(xh_1) = f(x) - f(xh_1)$$

and for every $n \geq 2$

$$\Delta_n f(x;h_1,\ldots,h_n) = \Delta_{n-1} f(x;h_1,\ldots,h_{n-1}) - \Delta_{n-1} f(xh_n;h_1,\ldots,h_{n-1}).$$

Definition 1.6. Let B a Banach $*$ algebra and A be a proper H^*-algebra. A linear mapping $f: B \to A$ is called a *positive A-functional* if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^* f(b_i^* b_j) a_j \geq 0$$

for all $b_1,\ldots,b_n$ in B and $a_1,\ldots,a_n$ in A.

Definition 1.7. Let S be a $*$-semigroup. Then a mapping $\varphi: S \to A$ is called *positive definite* if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^* \varphi(x_i^* x_j) a_j \geq 0$$

for all $x_1,\ldots,x_n$ in S and $a_1,\ldots,a_n$ in A.

Recall that a Borel measurable mapping $w: S \to \mathbb{R}_+$ (the set of nonnegative real numbers) with $w(xy) \leq w(x)w(y) \quad (x,y \in S)$ and such that w and $\frac{1}{w}$ are locally bounded (i.e., bounded on compact subsets of S) is called a *weight function* on S. A function $f: S \to \mathbb{C}$ is called w-bounded if there is a $k > 0$ such that $|f(x)| \leq kw(x)$, for all $x \in S$.

Recall also that $M(S,w)$ denotes the set of all complex, regular, signed measures μ (not necessarily bounded) of the form $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ where μ_i is a positive regular measure on S with $w \in L^1(S,\mu_i)$ $i = 1,2,3,4$ (see, for example [2], [7], [9]). Note that for an element $\mu \in M(S,w)$ and a Borel set B, $\mu(B)$ is well-defined whenever B is relatively compact. For every $\mu \in M(S,w)$, the equation

$$\int_S fd(w.\mu) = \int_S fwd\mu \qquad (f \in C_b(S)),$$

defines a measure $w.\mu \in M(S)$, the space of all bounded regular complex measures on S. With the norm

$$\|\mu\|_w = \|w.\mu\| \qquad (\mu \in M(S,w)),$$

where $\|w.\mu\|$ denotes the total variation of $w.\mu$, the space $M(S,w)$ defines a Banach lattice, and with the convolution product

$$(\mu * \nu)(f) = \int_S \int_S f(xy) d\mu(x) d\nu(y) \qquad (\mu,\nu \in, M(S,w),\ f \in C_{00}(S)), \tag{1}$$

where $C_{00}(S)$ denotes the set of all functions in $C_b(S)$ with compact support, defines a Banach algebra. From part (*iii*) of Theorem 4.6 of [7], we conclude that (1) also holds for every w-bounded Borel measurable function f on S.

We also recall (see, for example, [1], [6], [18]) that $M_a(S)$ (or $\widetilde{L}(S)$) denotes the set of all measures $\mu \in M(S)$ for which the mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ (where δ_x denotes the Dirac measure at x) from S into $M(S)$ are weakly continuous. As in [7], we can define $M_a(S,w)$ (or $\widetilde{L}(S,w)$) as the set of measures $\mu \in M(S,w)$ for which $w.\mu \in M_a(S)$. Then, $M_a(S,w)$ is a closed, two-sided L-ideal of $M(S,w)$. Finally, we call S a foundation semigroup if $\cup\{supp(\mu): \mu \in M_a(S)\}$ is dense in S. A mapping $\chi: S \to \mathbb{C}$ is called a semicharacter if $\chi(xy) = \chi(x)\chi(y)$ for all $x,y \in S$. We denote by Γ_w the set of all w-bounded continuous semicharacters on S, and by Γ_w^+ the set of nonnegative semicharacters in Γ_w. If S is commutative and foundation, then Γ_w is homomorphic to $\Delta(M_a(S,w))$ whenever Γ_w has the compact open topology and $\Delta(M_a(S,w))$ has the Gelfand topology. In particular; Γ_w is a locally compact Hausdorff space (see, Theorem 2.10 of [8]).

An operator-valued transformation $U: S \to \mathcal{L}(H)$ is called w-bounded (continuous, respectively) if for every $\xi,\eta \in H$ the map: $x \mapsto \langle U_x\xi,\eta\rangle$ is w-bounded (continuous, respectively). Finally if $U: S \to \mathcal{L}(H)$ is such that $U_{xy} = U_x U_y (x,y \in S)$, then U is called a *representation*. For further information on the representation theory of topological semigroups and *-algebras the reader is referred to [7].

2. Generalized Representations and Positive-Definite Functions on Weighted Foundation Semigroups

We start this section with the following result which is indeed a generalization of our earlier result (Theorem 4.4 of [7]).

Theorem 2.1. *Let S be a foundation ∗-semigroup with identity and with a Borel measurable weight function w such that $w(x^*) = w(x)\ (x \in S)$. Let T be a ∗-representation of $M_a(S,w)$ by bounded A-linear operators on a Hilbert module H over a proper H^*-algebra A such that for every $0 \neq \xi \in H$ there exists a measure $\mu \in M_a(S,w)$ such that $T_\mu \xi \neq 0$. Then there exists a unique w-bounded continuous ∗-representation V of S by A-linear operators on H such that*

$$(\eta, T_\mu \xi) = \int_S (\eta, V_x \xi) d\mu(x) \quad (\xi, \eta \in H, \mu \in M_a(S,w)). \tag{2}$$

Proof. Recall that by Theorem 1 of [11] H with the inner product $\langle .,. \rangle$ where $\langle \xi, \eta \rangle = tr(\eta, \xi)$ defines a Hilbert space and by Theorem 4 of [11], the adjoint operator T^* of T defines a bounded A-linear operator on H. So by Theorem 5.4 of [7] there exists a w-bounded continuous ∗-representation V of S by bounded operators on the Hilbert space $(H, \langle .,. \rangle)$ such that

$$\langle T_\mu \xi, \eta \rangle = \int_S \langle V_x \xi, \eta \rangle d\mu(x) \quad (\mu \in M_a(S,w), \xi, \eta \in H). \tag{3}$$

Now let $R(A)$ denote the space of the right centralizers of A. From Lemma 2 of [14] and Theorem 1 of [11] for every $U \in R(A)$ we have

$$trU(\eta, T_\mu \xi) = tr(U'\eta, T_\mu \xi) = \langle T_\mu \xi, U'\eta \rangle$$
$$= \int_S \langle V_x \xi, U'\eta \rangle d\mu(x) = \int_S trU(\eta, V_x \xi) d\mu(x).$$

So by Theorem 2 of [16]

$$(\eta, T_\mu \xi) = \int_S (\eta, V_x \xi) d\mu(x) \quad (\mu \in M_a(S,w), \xi, \eta \in H).$$

This proves formula (2).

We shall now use formula (3) and prove that if T_μ is A-linear for every $\mu \in M_a(S,w)$, then V_x is A-linear for every $x \in S$. To see this from (3) for every $\mu \in M_a(S,w)$, $\xi, \eta \in H$, and $a \in A$ we have

$$\int_S \langle \eta, V_x(\xi a) \rangle d\mu(x) = \langle \eta, T_\mu(\xi a) \rangle = tr(T_\mu(\xi a), \eta)$$
$$= tr((T_\mu \xi) a, \eta) = tr(T_\mu \xi, \eta a^*)$$
$$= \int_S \langle \eta a^*, V_x \xi \rangle d\mu(x) = \int_S \langle \eta, (V_x \xi) a \rangle d\mu(x).$$

Since both the mappings: $x \to \langle \eta, V_x(\xi a) \rangle$ and $x \to \langle \eta, (V_x \xi) a \rangle$ are w-bounded and continuous and S is a foundation semigroup, from Lemma 4.8 of [7] we conclude that $V_x(\xi a) = (V_x \xi) a\ (x \in S, a \in A)$.

The following result is indeed a generalization of our earlier version of Bochner's theorem [4; Theorem 4.2].

Theorem 2.2. (Generalized Bochner's theorem on foundation semigroups). *Let S be a commutative foundation topological ∗-semigroup with identity and with a Borel measurable weight function w. Let A be a proper H^*-algebra over a Hilbert module H. Then a mapping $\varphi : S \to \tau(A)$ is w-bounded continuous and positive definite if and only if there exists a unique positive A-valued measure λ_φ on Γ_w such that*

$$\varphi(x) = \int_{\Gamma_w^*} \chi(x) d\lambda_\varphi(\chi) \quad (x \in S).$$

Proof. Since φ is w-bounded and continuous, by Theorem 1 of [16] there exists a w-bounded weakly continuous ∗-representation V of S by bounded A-linear operators on a Hilbert A-module K with some $\xi_0 \in K$ such that $\varphi(x) = (\xi_0, T_x \xi_0)$ and $\|V_x\| \leq w(x)$ for every $x \in S$.

Using the integration theory on page 120 of [13] and Lemma 2 of the same reference, we conclude that the mapping $\Phi : M_a(S,w) \to \tau(A)$ given by

$$\Phi(\mu) = \int \varphi(x) d\mu(x) = \int_S (\xi_0, V_x \xi_0) d\mu(x) \ (\mu \in M_a S, w))$$

is well-defined. It is also easy to see the Φ defines a positive A-functional on the Banach ∗-algebra $M_a(S,w)$. Therefore, by Theorem 3 of [15] there exists a positive $\dot{\tau}(A)$-valued measure λ on $\Delta(M_a(S,w))$ such that

$$\Phi(\mu) = \int_{\Delta(M_a(S,w))} \hat{\mu}(\sigma) d\lambda(\sigma).$$

Using Theorem 2.10 of [7], we conclude that

$$\Phi(\mu) = \int_{\Gamma_w^*} \left(\int_S \chi(x) d\mu(x) \right) d\lambda(x) \quad (\mu \in M_a(S,w)).$$

By Fubini's theorem

$$\int_S \varphi(x) d\mu(x) = \int_S \left(\int_{\Gamma_w^*} \chi(x) d\lambda(\chi) \right) d\mu(x) (\mu \in M_a(S,w)).$$

Since both functions φ and $x \to \int_{\Gamma_w^*} \chi(x) d\lambda(\chi)$ are

w-bounded and weakly continuous and S is a foundation semigroup, we infer that

$$\varphi(x)=\int_{\Gamma_w^*}\chi(x)d\lambda(\chi)\quad (x\in S).$$

The uniqueness of λ follows in the same lines as those of Theorem 4.2 of [4].

3. Completely Monotone Functionals on Ordered Banach Algebras

Our starting point of this section is the following:

Theorem 3.1. *Let B be a commutative ordered Banach algebra with a bounded approximate identity* (e_α) *in* $P_1(B)$. *Let k be the set of all completely monotone functionals f in* B^* *such that* $\|f\|\le 1$. *Then K is a convex and weak*-compact subset of* A^*. *If* f *is an extreme point of K, then* $f(a)\ge 0$ *for all* $a\in P(B)$ *and* $f(ab)=f(a)f(b)$ *for all* $a,b\in B$.

Proof. It is clear that K is a convex and weak*-closed subset of the unit ball of B^* and so by the Banach Alaoglu theorem is weak*-compact. Let f be an extreme point of K. Then it is clear that $f(a)\ge 0$ for all $a\in P(B)$. Since $P_1(B)$ spans B, to prove that $f(ab)=f(a)f(b)$ for all $a,b\in B$, it suffices to show that $f(ab)=f(a)f(b)$ for all $a,b\in P_1(B)$. For every $a\in B$ we define $f_a\in B^*$ by $f_a(b)=f(ab)$ $(b\in B)$. It is easy to see that

$$\Delta_n(f-f_a)(b;b_1,\ldots,b_n)=\Delta_{n+1}f(b;b_1,\ldots,b_n,a),$$

for all $n\in\mathbb{Z}_+$, and $a,b,b_1,\ldots,b_n\in P_1(B)$. Thus, $f-f_a$ is also completely monotone. So

$$(f-f_a)(e_\alpha b)=\Delta_0(f-f_a)(e_\alpha b)\ge 0,$$

and

$$(f-f_a)(e_\alpha)-(f-f_a)(e_\alpha b)=\Delta_1(f-f_a)(e_\alpha;b)\ge 0,$$

for all $a,b\in P_1(B)$. From these two inequalities it follows that

$$0\le (f-f_a)(e_\alpha b)\le (f-f_a)(e_\alpha)=$$
$$f(e_\alpha)-f(ae_\alpha)\le 1-f(ae_\alpha)$$

for all α and all $a,b\in P_1(B)$. Since (e_α) is a bounded approximate identity for B, it follows that

$$0\le (f-f_a)(b)\le 1-f(a)\quad (a,b\in P_1(B)).\qquad (4)$$

Using the fact that f is completely monotone, we conclude that

$$0\le \Delta_0 f(ab)=f(ab)\quad (a,b\in P_1(B)),$$

and

$$0\le \Delta_1 f(a;b)=f(a)-f(ab)\quad (a,b\in P_1(B)).$$

Thus

$$0\le f(ab)\le f(a)\quad (a,b\in P_1(B)).\qquad (5)$$

We shall now consider three cases.

Case 1. $f(a)=0$. So by (5), $f(ab)=0$. Hence
$$f(ab)=0=f(a)f(b)\quad (a,b\in P_1(B)).$$
Case 2. $f(a)=1$. Then by (4),
$$f(b)=f(ab)(a,b\in P_1(B))\text{ and so}$$
$$f(ab)=f(b)=f(a)f(b)(a,b\in P_1(B)).$$
Case 3. $0<f(a)<1$. In this case we write

$$f=(1-f(a))\frac{f-f_a}{1-f(a)}+f(a)\frac{f_a}{f(a)}.$$

From (4) it follows that $(f-f_a)/(1-f(a))\in K$, and (5) implies that $f_a/f(a)$ also belongs to K. Since f is an extreme point of K, it follows that $f_a/f(a)=f$. So $f(ab)=f(a)f(b)$ for all $a,b\in P_1(B)$. This completes the proof.

Theorem 3.2. *Let B be a commutative ordered Banach algebra with a bounded approximate identity* (e_α) *in* $P_1(B)$. *Then a linear transformation* $U:B\to\mathcal{L}(H)$ *(H is a Hilbert space) is completely monotone if and only if there is a positive operator-valued measure E on* $\Delta_+(B)$ *such that*

$$\langle U_b\xi,\eta\rangle=\int_{\Delta_+(B)}\sigma(b)d\langle E_\sigma(.)\xi,\eta\rangle\quad (\xi,\eta\in H,b\in B).\ (6)$$

Moreover, U is a representation if and only if E is a spectral measure.

Proof. Let $U:B\to\mathcal{L}(H)$ be completely monotone. Without loss of generality, we may assume that $\|U_b\|\le\|b\|\,(b\in B)$. For every $\xi\in H$ with $\|\xi\|=1$ we define the linear functional L_ξ on B by

$$L_\xi(b)=\langle U_b\xi,\xi\rangle\quad (b\in B).$$

It is clear that L_ξ defines a completely monotone functional on B with $\|L_\xi\|\le 1$. By the integral form of the Krein-Milman theorem [10; p. 6] and Theorem 3.1, there exists a unique regular probability measure $\mu_{\xi,\xi}$ on

$\Delta_+(B)$ such that

$$L_\xi(b)=\int_{\Delta_+(B)}\sigma(b)d\mu_{\xi,\xi}(\sigma)\quad (b\in B).$$

So if $0\neq\xi\in H$ is arbitrary, then there exists a unique positive regular measure $\mu_{\xi,\xi}$ with $\|\mu_{\xi,\xi}\|\le\|\xi\|^2$ and

$$\langle U_b\xi,\xi\rangle=\int_{\Delta_+(B)}\sigma(b)d\mu_{\xi,\xi}(\sigma)\quad (b\in B).$$

By the polarization identity for every $\xi,\eta\in H$ and $b\in B$ we have

$$\langle U_b\xi,\eta\rangle=\frac{1}{4}(\langle U_b(\xi+\eta),\xi+\eta\rangle-\langle U_b(\xi-\eta),\xi-\eta\rangle$$
$$+i\langle U_b(\xi+i\eta),\xi+i\eta\rangle-i\langle U_b(\xi-i\eta),\xi-i\eta\rangle).$$

Thus

$$\langle U_b\xi,\eta\rangle=\int_{\Delta_+(B)}\sigma(b)d\mu_{\xi,\eta}(\sigma)\quad (b\in B,\xi,\eta\in H),$$

where

$$\mu_{\xi,\eta}=\frac{1}{4}\left(\mu_{\xi+\eta,\xi+\eta}-\mu_{\xi-\eta,\xi-\eta}+i\mu_{\xi+i\eta,\xi+i\eta}-i\mu_{\xi+i\eta,\xi+i\eta}\right).$$

Now let $\mathcal{B}(\Delta_+(B))$ denote the σ-algebra of all Borel subsets of $\Delta_+(B)$. Define the operator-valued measure E on $\mathcal{B}(\Delta_+(B))$ by

$$\langle E(M)\xi,\eta\rangle=\mu_{\xi,\eta}(M)\quad (\xi,\eta\in H, M\in\mathcal{B}(\Delta_+(B))).$$

It is easy to see that E is positive, in the sense that $\langle E(M)\xi,\xi\rangle\ge 0$ for all $\xi\in H$ and $M\in\mathcal{B}(\Delta_+(B))$. Moreover,

$$\langle U_b\xi,\eta\rangle=\int_{\Delta_+(B)}\sigma(b)d\langle E_\sigma(.)\xi,\eta\rangle\quad (b\in B,\xi,\eta\in H).$$

For simplicity, we abbreviate this equality as

$$U_b=\int_{\Delta_+(B)}\sigma(b)dE_\sigma\quad (b\in B).$$

Now for every $b\in B$ we denote by $\hat{b}$ the restriction of the Gelfand transform of b to $\Delta_+(B)$, that is $\hat{b}(\sigma)=\sigma(b)$ for all $\sigma\in\Delta_+(B)$. Since by the Gelfand representation theorem $\mathcal{P}=\{\hat{b}:b\in B\}$ separates the points of $\Delta_+(B)$, from the Stone-Weierstrass theorem it follows that it is dense in $C_0(\Delta_+(B))$, the space of all continuous complex-valued functions on $\Delta_+(B)$ vanishing at infinity. Now if U is multiplicative, then for every $a,b\in B$ we have

$$\int_{\Delta_+(B)}\hat{a}(\sigma)\hat{b}(\sigma)dE_\sigma=\int_{\Delta_+(B)}\widehat{ab}(\sigma)dE_\sigma=U_{ab}=U_aU_b$$
$$=\int_{\Delta_+(B)}\hat{a}(\sigma)dE_\sigma\int_{\Delta_+(B)}\hat{b}(\sigma)dE_\sigma.$$

Since for every fixed $b\in B$, each of the functions $\hat{a}\mapsto\int_{\Delta_+(B)}\widehat{ab}(\sigma)dE_\sigma$ and $\hat{a}\mapsto\int_{\Delta_+(B)}\hat{a}(\sigma)dE_\sigma$ $(\Delta_+(B))$, $\int_{\Delta_+(B)}\hat{b}(\sigma)dE_\sigma\,(\hat{a}\in\mathcal{P})$ are bounded and linear on $\mathcal{P}$, and $\mathcal{P}$ is dense in $C_0(\Delta_+(B))$, then for every Borel subset M of $\Delta_+(B)$ we have

$$\int_{\Delta_+(B)}1_M(\sigma)\hat{a}(\sigma)dE_\sigma=$$
$$\int_{\Delta_+(B)}1_M(\sigma)dE_\sigma\int_{\Delta_+(B)}\hat{a}(\sigma)dE_\sigma$$

where 1_M denotes the characteristic function of the set M. A similar argument shows that for every two Borel subsets M and N of $\Delta_+(B)$

$$\int_{\Delta_+(B)}1_M(\sigma)1_N(\sigma)dE_\sigma=$$
$$\int_{\Delta_+(B)}1_M(\sigma)dE_\sigma\int_{\Delta_+(B)}1_N(\sigma)dE_\sigma.$$

That is $E(M\cap N)=E(M)E(N)$. So E is a spectral measure on $\Delta_+(B)$. The proof is now complete.

The following theorem gives a characterization of the completely monotone functionals on commutative ordered Banach algebras.

Theorem 3.3. *Let B be a commutative ordered Banach algebra with a bounded approximate identity in $P_1(B)$. Then a bounded linear mapping U of B into a proper H^*-algebra A is completely monotone if and only if there is a positive $\tau(A)$-valued measure E on $\Delta_+(B)$ such that*

$$(\xi,\eta U(b))=\int_{\Delta_+(B)}\sigma(b)d(\xi,\eta E_\sigma(.))\,(b\in B,\xi,\eta\in H).$$

Moreover, U is a positive homomorphism if and only if E is a generalized spectral measure.

Proof. For every $\xi\in H$ with $tr(\xi,\xi)=1$ we define

$$L_\xi(b)=tr(\xi,\xi U(b))=\langle\xi U(b),\xi\rangle\quad (b\in B).$$

From

$$\Delta_n(\xi,\xi U(b;b_1,\dots b_n))=(\xi,\xi)\Delta_nU(b;b_1,\dots,b_n)$$
$$(b;b_1,\dots b_n\in B,\ n\in\mathbb{Z}_+)$$

and the fact that U is bounded and completely monotone we conclude that L_ξ defines a completely linear functional on B. So by Theorem 3.2 there exists an operator-valued measure E by bounded operators on the Hilbert space $(H,\langle .,.\rangle)$ such that

$$L_\xi(b)=\langle \xi U(b),\eta\rangle=\int_{\Delta_+(B)}\sigma(b)d\langle E_\sigma(.)\xi,\eta\rangle \quad (b\in B).$$

For every $T\in R(A)$ by Lemma 2 of [14] we have

$$trT(\eta,\xi U(b))=tr(T'\eta,\xi U(b))=\langle \xi U(b),T'\eta\rangle$$

$$=\int_{\Delta_+(B)}\sigma(b)d\langle E_\sigma(.)\xi,T'\eta\rangle$$

$$=\int_{\Delta_+(B)}\sigma(b)d\ trT(\eta,E_\sigma(.)\xi).$$

Now it is easily seen that the mapping: $\Delta_+(B)\to\tau(A)$ given by: $M\to(\eta,E(M)\xi)$ $(M\in\mathcal{B}(\Delta_+(B))$ defines a $\tau(A)$-valued measure on $\Delta_+(B)$. Therefore, by Lemma 2 of [14] and Theorem 2 of [16] we have

$$(\eta,\xi U(b))=\int_{\Delta_+(B)}\sigma(b)d(\eta,E_\sigma(.)\xi) \quad (b\in B).$$

Thus, the proof is complete.

We are now in a position to state and prove the main result of this paper. Note that if H is a right Hilbert module over a proper H^*-algebra A, then a mapping $T:S\to A$ is called w-bounded and continuous if for every $\xi,\eta\in H$ the mapping $x\mapsto tr(\xi,\eta T_x)$ is a w-bounded continuous complex-valued function on S.

Theorem 3.4. *Let S be a commutative foundation semigroup with identity and with a Borel measurable weight function w continuous at the identity. Let H be a Hilbert module over a proper H^*-algebra A. Then a mapping $T:S\to A$ is w-bounded continuous and completely monotone if and only if there exists a unique positive A-valued measure E on Γ_w^+ such that*

$$(\xi,\eta T_x)=\int_{\Gamma_w^+}\chi(x)d(\xi,E_\chi(.)\eta) \quad (x\in S,\xi,\eta\in H).$$

T is a homomorphism if and only if E is a generalized A-valued spectral measure.

Proof. From the continuity of w at the identity of S it follows that $M_a(S,w)$ has a bounded approximate identity in $P_1(M_a(S,w))$ (see [9]). It is also easy to see that the equation

$$(\xi,\eta U(\mu))=\int_S(\xi,\eta T_x)d\mu(x) \quad (\mu\in M_a(S,w),\xi,\eta\in H)$$

defines a completely monotone A-valued bounded functional on the ordered Banach algebra $M_a(S,w)$. Therefore, by Theorem 3.3 there exists a unique positive A-valued measure E on $\Delta_+(M_a(S,w))$ such that

$$(\xi,\eta U(\mu))=\int_{\Delta_+(M_a(S,w))}\hat{\mu}(\chi)d(\xi,E_\chi(.)\eta)$$

$$(\xi,\eta\in H,\ \mu\in M_a(S,w)).$$

Now an application of this equality and Theorem 2.10 of [8] with the aid of Fubini's theorem gives

$$(\xi,\eta U(\mu))=\int_{\Gamma_w^+}\int_S\chi(x)d\mu(x)d(\xi,E_\chi(.)\eta)$$

$$=\int_S\int_{\Gamma_w^+}\chi(x)d(\xi,E_\chi(.)\eta)d\mu(x).$$

Now since both mappings $x\mapsto\int_{\Gamma_w^+}\chi(x)d\langle\xi,E_\chi(.)\eta\rangle$ and $x\mapsto\langle\xi,\eta T_x\rangle$ are w-bounded and continuous and S is also a foundation semigroup, we conclude that

$$(\xi,\eta T_x)=\int_{\Gamma_w^+}\chi(x)d(\xi,E_\chi(.)\eta) \quad (\xi,\eta\in H,\ x\in S).$$

Remark. The following example shows that the conclusion of the preceding theorem is not valid in general for non-foundation semigroups.

Example 3.5. Let $S=[0,1]$. Then with the usual topology of the real line and the multiplication $xy=\min(x,y)(x,y\in S)$ S defines a non-foundation semigroup. If we choose $w=1$ on S, then $\Gamma_w^+=\{1\}$, where 1 denotes the function which is identically one on S. It is clear that the mapping $T:S\to\mathcal{L}(L^2(S,m))$ (m denotes the Lebesgue measure on [0,1]) given by

$$T_x f=\hat{x}f \qquad (x\in S,\ f\in L^2(S,m)),$$

where $\hat{x}$ denotes the characteristics function on [0,x], defines a completely monotone operator-valued transformation of S by operators on the Hilbert module $L^2(S,m)$ (see [3]). If the formula (6) is valid for T, then we arrive at the contradiction that $T_x=I$ for every x in S, where I denotes the identity operator on $L^2(S,m)$.

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