HYPOTHESIS TESTING FOR AN EXCHANGEABLE NORMAL DISTRIBUTION

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Abstract

Consider an exchangeable normal vector with parameters μ, σ^2, and ρ. On the basis of a vector observation some tests about these parameters are found and their properties are discussed. A simulation study for these tests and a few nonparametric tests are presented. Some advantages and disadvantages of these tests are discussed and a few applications are given.

1. Introduction

Statistical studies are often based on independent and identically distributed (IID) random variables. In applications we may not have such strong assumptions on the observations. A weaker assumption is exchangeability. Exchangeable random variables were first introduced by de Finetti [5] and then considered by many researchers, for example, Chow & Teicker [4], de Finetti [6,7,8], Feller [10], Fürst [12], and Koch & Spizzichino [14].

This work is concerned with hypothesis testing for an exchangeable normal distribution. The random vector $\mathbf{X}=(X_1,\ldots, X_p)'$ is said to have an exchangeable normal distribution if its distribution is multivariate normal with the following mean vector and variance-covariance matrix

$$\begin{pmatrix}\mu\\ \mu\\ \cdot\\ \mu\end{pmatrix}_{p\times 1}, -\infty<\mu<\infty,\ \sigma^2\begin{pmatrix}1 & \rho & \ldots & \rho\\ \rho & 1 & \ldots & \rho\\ \cdot & \cdot & \ldots & \cdot\\ \rho & \rho & \ldots & 1\end{pmatrix}_{p\times p}, \sigma>0, \rho\in[0,1).$$

Keywords: Exchangeable normal distribution; Power function; Robust test; Test of randomness; Uniformly most powerful test; Uniformly most powerful unbiased test

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(see, e.g., Tong [21], page 112). We denote this exchangeable normal distribution with three parameters μ, σ^2, and ρ by $EN_p(\mu, \sigma^2, \rho)$. It is clear that $(X_1,\ldots, X_p)$ and $(X_{i_1},\ldots, X_{i_p})$ are identical in distribution for any permutation $\{i_1,\ldots,i_p\}$ of $\{1,\ldots,p\}$.

Some statisticians have worked on this distribution. For example, Rao [19] has a t-test for μ, McElroy [17] considers a regression study with exchangeable normal errors, and Arnold [1] extends this study to linear models.

In Section 2, we study some tests about the parameters of an exchangeable normal distribution, and we plot their power functions. Section 3 is concerned with a simulation study and a comparison of these tests with a few nonparametric tests. In Section 4 a few applications are given for these tests.

2. Hypothesis Testing

In this section we introduce some intuitively test functions for testing the parameters of an exchangeable normal vector **X**. We also point out some restrictions on these tests and find the best ones.

First we study two tests for ρ. Suppose we want to test $\begin{cases}H_0:\rho=0\\ H_1:\rho>0\end{cases}$, when μ is known. It can be proved that

$$\frac{\bar{X}-\mu}{S/\sqrt{p}}\sqrt{\frac{(1-\rho)}{1+(p-1)\rho}} \stackrel{d}{=} T_{p-1},$$

where $\stackrel{d}{=}$ denotes equality in distribution, $\bar{X}$ and S^2 denotes the sample mean and variance respectively, and $T_{p\text{-}1}$ denotes the t-variable with p-1 degrees of freedom (see Rao [19] page 197). Under $H_0:\rho=0$, we have $\frac{\bar{X}-\mu}{S/\sqrt{p}} \stackrel{d}{=} T_{p-1}$. If ρ is close to 1 then $\frac{(1-\rho)}{1+(p-1)\rho}$ is close to 0. Therefore, we reject H_0 if $\frac{\bar{x}-\mu}{s/\sqrt{p}} > k_1$ or k_2, where $k_1 = t_{p-1,1-\frac{\alpha}{2}}$, and $k_2 = -k_1$, and $P(T_{p-1} > t_{p-1,1-\frac{\alpha}{2}}) = \frac{\alpha}{2}$.

Now, consider the case that σ^2 is known but μ is unknown. It can be proved that

$$\frac{(p-1)S^2}{(1-\rho)\sigma^2} \stackrel{d}{=} \chi^2_{p-1},$$

where χ^2_{p-1} denotes the chi-square distribution with p-1 degrees of freedom (see Rao [19] page 197). Under $H_0:\rho=0$, we have $(p-1)S^2/\sigma^2 \stackrel{d}{=} \chi^2_{p-1}$. If ρ is close to 1 then (1-ρ) is close to 0. Thus, we reject H_0, if $(p-1)s^2/\sigma^2 < \chi^2_{p-1,\alpha}$, where $P_{\rho=0}((p-1)S^2/\sigma^2 < \chi^2_{p-1,\alpha}) = \alpha$.

Table 1 shows intuitively test functions for parameters of an exchangeable normal vector and Figures 1-4 show their power functions (the graphs and the computations are prepared by S-PLUS). We use the following abbreviation in this table, K: known, P: parameter, Prop: property, UMP: uniformly most powerful, UMPU: UMP unbiased.

Note that, there is no test for anyone of μ, σ^2, and ρ, when two of them are unknown. A main reason for this, due to the fact that dimension of minimal sufficient statistic is less than the dimension of parameter space (see Remark 2.1).

In the following theorem we prove that some of the test functions in Table 1 are the best.

Theorem 2.1. If **X**=($X_1,\ldots, X_p$)′ has the distribution $EN_p(\mu, \sigma^2, \rho)$, then the test functions in Table 1 follows the properties in the last column of this table.

Proof. *Hypothesis testing for* ρ. Let μ be known. Without loss of generally assume that μ =0. First consider the case p=2. In this case the joint density of **X**=(X_1, X_2)′ is given by

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}) &= \left(2\pi\sigma^2\sqrt{1-\rho^2}\right)^{-1} \times \\ &\quad \exp\left\{\frac{-1}{2(1-\rho^2)\sigma^2}\left(x_1^2+x_2^2-2\rho x_1x_2\right)\right\} \\ &= \left(2\pi\sigma^2\sqrt{1-\rho^2}\right)^{-1} \times \\ &\quad \exp\left\{\frac{\rho}{(1-\rho^2)\sigma^2}x_1x_2 - \frac{1}{2(1-\rho^2)\sigma^2}\left(x_1^2+x_2^2\right)\right\} \\ &= k(\theta_1,\theta_2)\exp\{\theta_1t_1+\theta_2t_2\}, \end{aligned}$$

where $\theta_1 = \frac{\rho}{(1-\rho^2)\sigma^2}$, $t_1 = x_1x_2$, $\theta_2 = \frac{-1}{2(1-\rho^2)\sigma^2}$, $t_2 = x_1^2 + x_2^2$, and $k(\theta_1,\theta_2)$ is a function of θ_1, θ_2.

Now we can apply Theorem 3, page 147 of Lehmann [15]. The test function $\phi(t_1,t_2)$ given by $\phi(t_1,t_2) = \begin{cases} 1 & t_1 > c(t_2) \\ 0 & t_1 < c(t_2) \end{cases}$ is an UMPU test for testing $\begin{cases} H_0^*:\theta_1 = 0 \\ H_1^*:\theta_1 > 0 \end{cases}$, where $c(t_2)$ is so chosen that $P_{\theta_1=0}(T_1 > c(T_2)\,|\,T_2 = t_2) = \alpha$. But $\begin{cases} H_0^*:\theta_1 = 0 \\ H_1^*:\theta_1 > 0 \end{cases}$ is equivalent to $\begin{cases} H_0:\rho = 0 \\ H_1:\rho > 0 \end{cases}$ and on the boundary of H_0^*, H_1^* (or H_0, H_1) i.e. on θ_1=0 (or ρ=0), T_2 is a complete sufficient statistic for σ^2. If we define

$$\begin{aligned} T' &= \frac{2T_1+T_2}{T_2} = \frac{2X_1X_2+X_1^2+X_2^2}{X_1^2+X_2^2} \\ &= \frac{(X_1+X_2)^2}{X_1^2+X_2^2} = \frac{\left(\frac{X_1}{\sigma}+\frac{X_2}{\sigma}\right)^2}{\frac{X_1^2+X_2^2}{\sigma^2}}, \end{aligned}$$

then T' is an ancillary and as a result independent from T_2. Therefore,

$$\begin{aligned} P_{\theta_1=0}(T_1 > c(T_2)\,|\,T_2 = t_2) &= P_{\theta_1=0}(T' > c_1(T_2)\,|\,T_2 = t_2) \\ &= P_{\theta_1=0}(T' > c_2), \end{aligned}$$

where $c_1(T_2) = \frac{2c(T_2)+T_2}{T_2}$ and $c_2 = c_1(t_2)$ is a constant to be determined for a given α. Hence,

$$\phi(t_1,t_2) = \begin{cases} 1 & \frac{2t_1+t_2}{t_2} > c_2 \\ 0 & \frac{2t_1+t_2}{t_2} < c_2, \end{cases}$$

Table 1. Intuitively test functions for hypothesis testing for an $EN_p(\mu,\ \sigma^2,\ \rho)$

Hypotheses	**K. P.**	**Test function**	**Prop.**
$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu \neq \mu_0 \end{cases}$	σ^2, ρ	$\phi_1(\mathbf{x}) = \begin{cases} 1 & \sqrt{p}\lvert\bar{x}-\mu_0\rvert / \sigma\sqrt{1+(p-1)\rho} > -z_{\alpha/2} \\ 0 & \sqrt{p}\lvert\bar{x}-\mu_0\rvert / \sigma\sqrt{1+(p-1)\rho} < -z_{\alpha/2} \end{cases}$	UMPU
	ρ	$\phi_2(\mathbf{x}) = \begin{cases} 1 & \dfrac{\sqrt{p(1-\rho)}\lvert\bar{x}-\mu_0\rvert}{s\sqrt{1+(p-1)\rho}} > -t_{p-1,\alpha/2} \\ 0 & \dfrac{\sqrt{p(1-\rho)}\lvert\bar{x}-\mu_0\rvert}{s\sqrt{1+(p-1)\rho}} < -t_{p-1,\alpha/2} \end{cases}$	UMPU
	σ^2	The test can be done by some approximations	
		?	
$\begin{cases} H_0 : \sigma^2 = \sigma_0^2 \\ H_1 : \sigma^2 < \sigma_0^2 \end{cases}$	μ, ρ	$\phi_3(\mathbf{x}) = \begin{cases} 1 & w = \dfrac{p(\bar{x}-\mu)^2}{\sigma_0^2(1+(p-1)\rho)} + \dfrac{(p-1)s^2}{\sigma_0^2(1-\rho)} < \chi^2_{p,\alpha} \\ 0 & w = \dfrac{p(\bar{x}-\mu)^2}{\sigma_0^2(1+(p-1)\rho)} + \dfrac{(p-1)s^2}{\sigma_0^2(1-\rho)} > \chi^2_{p,\alpha} \end{cases}$	UMP
	ρ	$\phi_4(\mathbf{x}) = \begin{cases} 1 & (p-1)s^2/\sigma_0^2(1-p) < \chi^2_{p-1,\alpha} \\ 0 & (p-1)s^2/\sigma_0^2(1-p) > \chi^2_{p-1,\alpha} \end{cases}$	UMPU
	μ	The test can be done by some approximations	
		?	
$\begin{cases} H_0 : \rho = 0 \\ H_1 : \rho > 0 \end{cases}$	μ, σ^2	$\phi_5(\mathbf{x}) = \begin{cases} 1 & \sqrt{p}\lvert\bar{x}-\mu\rvert / \sigma > -z_{\alpha/2} \\ 0 & \sqrt{p}\lvert\bar{x}-\mu\rvert / \sigma < -z_{\alpha/2} \end{cases}$	
	μ, σ^2	$\phi_6(\mathbf{x}) = \begin{cases} 1 & \sum_{i=1}^{p}(x_i-\mu)^2/\sigma^2 < \chi^2_{p,\alpha} \\ 0 & \sum_{i=1}^{p}(x_i-\mu)^2/\sigma^2 > \chi^2_{p,\alpha} \end{cases}$	
	σ^2	$\phi_7(\mathbf{x}) = \begin{cases} 1 & (p-1)s^2/\sigma^2 < \chi^2_{p-1,\alpha} \\ 0 & (p-1)s^2/\sigma^2 > \chi^2_{p-1,\alpha} \end{cases}$	
	μ	$\phi_8(\mathbf{x}) = \begin{cases} 1 & \sqrt{p}\lvert\bar{x}-\mu\rvert / s > -t_{p-1,\alpha/2} \\ 0 & \sqrt{p}\lvert\bar{x}-\mu\rvert / s < -t_{p-1,\alpha/2} \end{cases}$	UMPU
		?	
$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu > \mu_0 \end{cases}$	σ^2, ρ	$\phi_1^1(\mathbf{x}) = \begin{cases} 1 & \sqrt{p}(\bar{x}-\mu_0)/\sigma\sqrt{1+(p-1)\rho} > -z_\alpha \\ 0 & \sqrt{p}(\bar{x}-\mu_0)/\sigma\sqrt{1+(p-1)\rho} < -z_\alpha \end{cases}$	UMP
$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu < \mu_0 \end{cases}$	σ^2, ρ	$\phi_1^2(\mathbf{x}) = \begin{cases} 1 & \sqrt{p}(\bar{x}-\mu_0)/\sigma\sqrt{1+(p-1)\rho} < z_\alpha \\ 0 & \sqrt{p}(\bar{x}-\mu_0)/\sigma\sqrt{1+(p-1)\rho} > z_\alpha \end{cases}$	UMP
$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu > \mu_0 \end{cases}$	ρ	$\phi_2^1(\mathbf{x}) = \begin{cases} 1 & \dfrac{\sqrt{p(1-\rho)}(\bar{x}-\mu_0)}{s\sqrt{1+(p-1)\rho}} > -t_{p-1,\alpha} \\ 0 & \dfrac{\sqrt{p(1-\rho)}(\bar{x}-\mu_0)}{s\sqrt{1+(p-1)\rho}} < -t_{p-1,\alpha} \end{cases}$	UMPU
$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu < \mu_0 \end{cases}$	ρ	$\phi_2^2(\mathbf{x}) = \begin{cases} 1 & \dfrac{\sqrt{p(1-\rho)}(\bar{x}-\mu_0)}{\sqrt{1+(p-1)\rho}} < t_{p-1,\alpha} \\ 0 & \dfrac{\sqrt{p(1-\rho)}(\bar{x}-\mu_0)}{\sqrt{1+(p-1)\rho}} > t_{p-1,\alpha} \end{cases}$	UMPU
$\begin{cases} H_0 : \sigma^2 = \sigma_0^2 \\ H_1 : \sigma^2 > \sigma_0^2 \end{cases}$	μ, ρ	$\phi_3^1(\mathbf{x}) = \begin{cases} 1 & \dfrac{p(\bar{x}-\mu)^2}{\sigma_0^2(1+(p-1)\rho)} + \dfrac{(p-1)s^2}{\sigma_0^2(1-\rho)} > \chi^2_{p,1-\alpha} \\ 0 & \dfrac{p(\bar{x}-\mu)^2}{\sigma_0^2(1+(p-1)\rho)} + \dfrac{(p-1)s^2}{\sigma_0^2(1-\rho)} < \chi^2_{p,1-\alpha} \end{cases}$	UMP

Table 1. Continued

Hypotheses	K. P.	Test function	Prop.
$\begin{cases} H_0: \sigma^2 = \sigma_0^2 \\ H_1: \sigma^2 \neq \sigma_0^2 \end{cases}$	μ, ρ	$\phi_3^2(\mathbf{x}) = \begin{cases} 1 & w > \chi^2_{p,1-\frac{\alpha}{2}}, \text{ or } < \chi^2_{p,\frac{\alpha}{2}} \\ 0 & \chi^2_{p,\frac{\alpha}{2}} < w < \chi^2_{p,1-\frac{\alpha}{2}} \end{cases}$	
$\begin{cases} H_0: \sigma^2 = \sigma_0^2 \\ H_1: \sigma^2 > \sigma_0^2 \end{cases}$	ρ	$\phi_4^1(\mathbf{x}) = \begin{cases} 1 & (p-1)s^2/\sigma_0^2(1-\rho) > \chi^2_{p-1,1-\alpha} \\ 0 & (p-1)s^2/\sigma_0^2(1-\rho) < \chi^2_{p-1,1-\alpha} \end{cases}$	UMPU
$\begin{cases} H_0: \sigma^2 = \sigma_0^2 \\ H_1: \sigma^2 \neq \sigma_0^2 \end{cases}$	ρ	$\phi_4^2(\mathbf{x}) = \begin{cases} 1 & \frac{(p-1)s^2}{\sigma_0^2(1-\rho)} > \chi^2_{p-1,1-\frac{\alpha}{2}}, \text{ or } < \chi^2_{p-1,\frac{\alpha}{2}} \\ 0 & \chi^2_{p-1,\frac{\alpha}{2}} < \frac{(p-1)s^2}{\sigma_0^2(1-\rho)} < \chi^2_{p-1,1-\frac{\alpha}{2}} \end{cases}$	

where c_2 may be chosen so that $P_{\theta_1=0}(T' > c_2) = \alpha$. In fact, this test is the usual *t*-test, more often written in the form of

$$\phi(\mathrm{x}) = \begin{cases} 1 & \dfrac{|\bar{x}|}{s/\sqrt{p}} > c' \\ 0 & \dfrac{|\bar{x}|}{s/\sqrt{p}} < c', \end{cases}$$

where p=2, $|\bar{x}| = \dfrac{\sqrt{2t_1 + t_2}}{p}$, $s^2 = \sum_{i=1}^p (x_i - \bar{x})^2/(p-1) = (t_2 - p\bar{x}^2)/(p-1)$ (see Ferguson [11] page 230). With $c' = t(p-1;1-\frac{\alpha}{2})$, the test function $\phi(\mathbf{x})$ is an UMPU size-α test for testing $\begin{cases} H_0: \rho = 0 \\ H_1: \rho > 0 \end{cases}$.

When p>2, the proof is similar to the above proof. In this case, by some simple algebraic calculations or using an orthogonal transformation we can show that the density of $\mathbf{X}=(X_1,\dots, X_p)'$ is

$$f_{\mathbf{X}}(\mathbf{x}) = k_p(\rho,\sigma^2)\exp\left\{-\frac{1}{2}\left[\frac{\sum_{i=1}^p x_i^2}{\sigma^2(1-\rho)} - \frac{\rho\left(\sum_{i=1}^p x_i\right)^2}{\sigma^2(1+(p-1)\rho)(1-\rho)}\right]\right\}$$
$$= k_p(\rho,\sigma^2)\exp\left\{\frac{p\rho(\sqrt{p}\bar{x})^2}{2\sigma^2(1+(p-1)\rho)(1-\rho)} - \frac{\sum_{i=1}^p x_i^2}{2\sigma^2(1-\rho)}\right\},$$

where

$$k_p(\rho,\sigma^2) = (\sqrt{2\pi}\sigma)^{-p}(1-p)^{-(p-1)/2}(1+(p-1)\rho)^{-1/2}$$

(see Tong [21] page 112, formula (5.3.8)′ which contains an error). Note that, $(\sqrt{p}\bar{x})^2 = (2\sum_{i<j} x_i x_j + \sum_{i=1}^p x_i^2)/p$.

Therefore,

$$f_{\mathbf{X}}(\mathbf{x}) = k'_p(\theta_1,\theta_2)\exp\{\theta_1 t_1, \theta_2 t_2\}, \qquad (2.1)$$

where $\theta_1 = \rho/(\sigma^2(1+(p-1)\rho)(1-\rho))$, $t_1 = \sum_{i<j} x_i x_j$, $t_2 = \sum_{i=1}^p x_i^2$, $k'_p(\theta_1,\theta_2)$ is a function of θ_1, θ_2, and θ_2 can be determined. The rest of this case is similar to the case p=2. Therefore, ϕ_8 is UMPU.

Hypothesis Testing for μ. Note that $\bar{X}$ and $(\bar{X}, S^2)$ are minimal sufficient statistics for μ and (μ, σ^2) when (σ^2, ρ) are known, respectively. It is known that $\bar{X} \overset{d}{=} N(\mu,(1+(p-1)\rho)/p)$, and $(p-1)S^2/((1-\rho)\sigma^2) \overset{d}{=} \chi^2_{p-1}$ are independent. Therefore, the properties of the tests ϕ_1, ϕ_1^1, ϕ_1^2, ϕ_2, ϕ_2^1 and ϕ_2^2 can be proved immediately (see e.g., Lehmann [15] page 192).

Hypothesis Testing for σ^2. Let (μ, ρ) be known. Fix σ^2 under H_1 and apply the Neyman-Pearson lemma (see also Hypothesis testing for ρ). Then we have the properties of ϕ_3, and ϕ_3^1. The proof for the properties of ϕ_4, and ϕ_4^1 is similar to the test functions of μ.

Remark 2.1. If μ and σ^2 are both unknown we have trivial UMPU test for ρ. To prove this fact we observe that

$$f_{\mathbf{X}}(\mathbf{x}) = q_p(\theta_1,\theta_2,\theta_3)\exp\{\theta_1 t_1 + \theta_2 t_2 + \theta_3 t_3\},$$

where

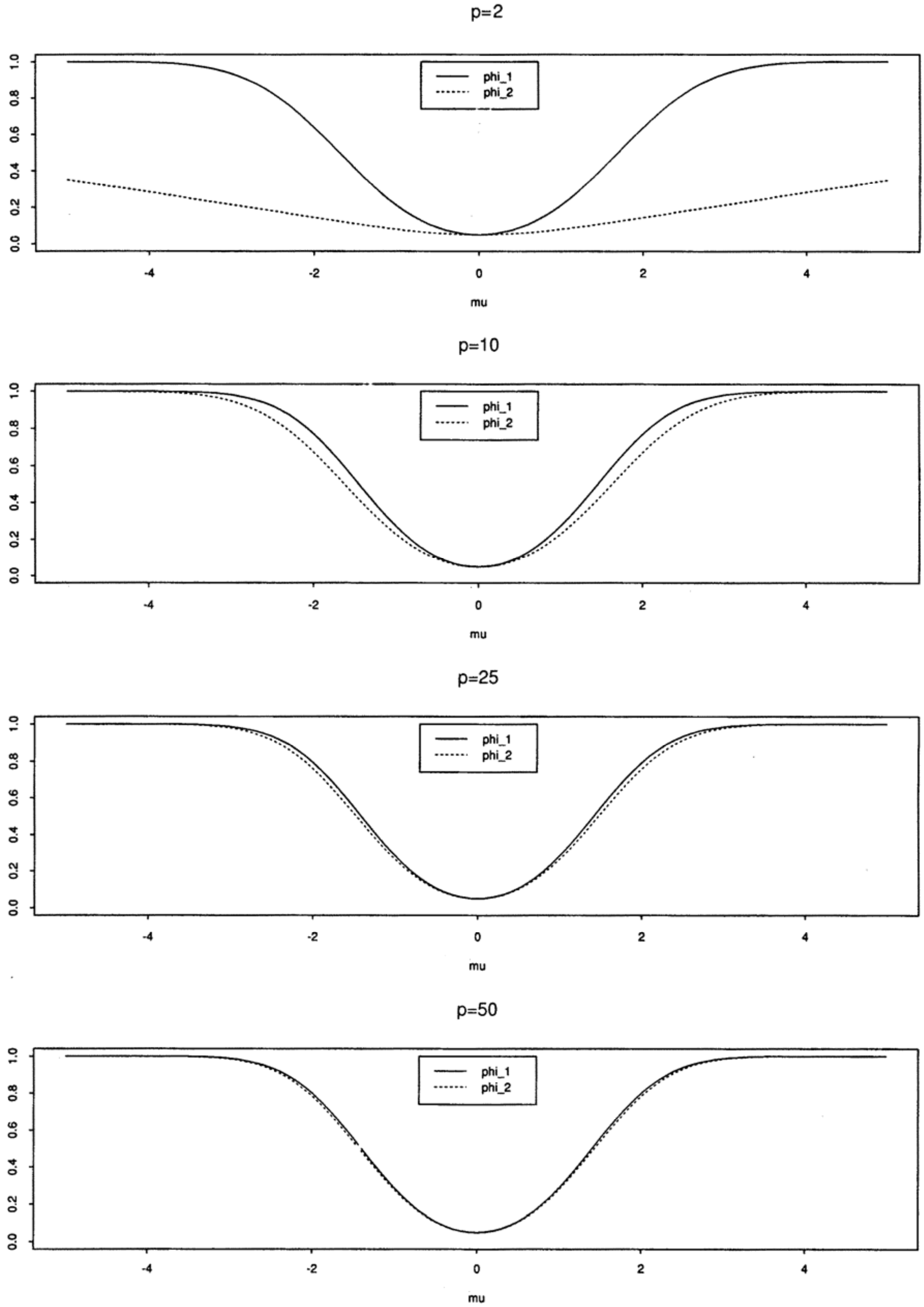

Figure 1. Power functions for the tests ϕ_1, and ϕ_2, where $\alpha = 0.05$, $\mu_0 = 0$, $\sigma^2 = 1$, and $\rho = 0.5$, for p = 2, 10, 25, 50.

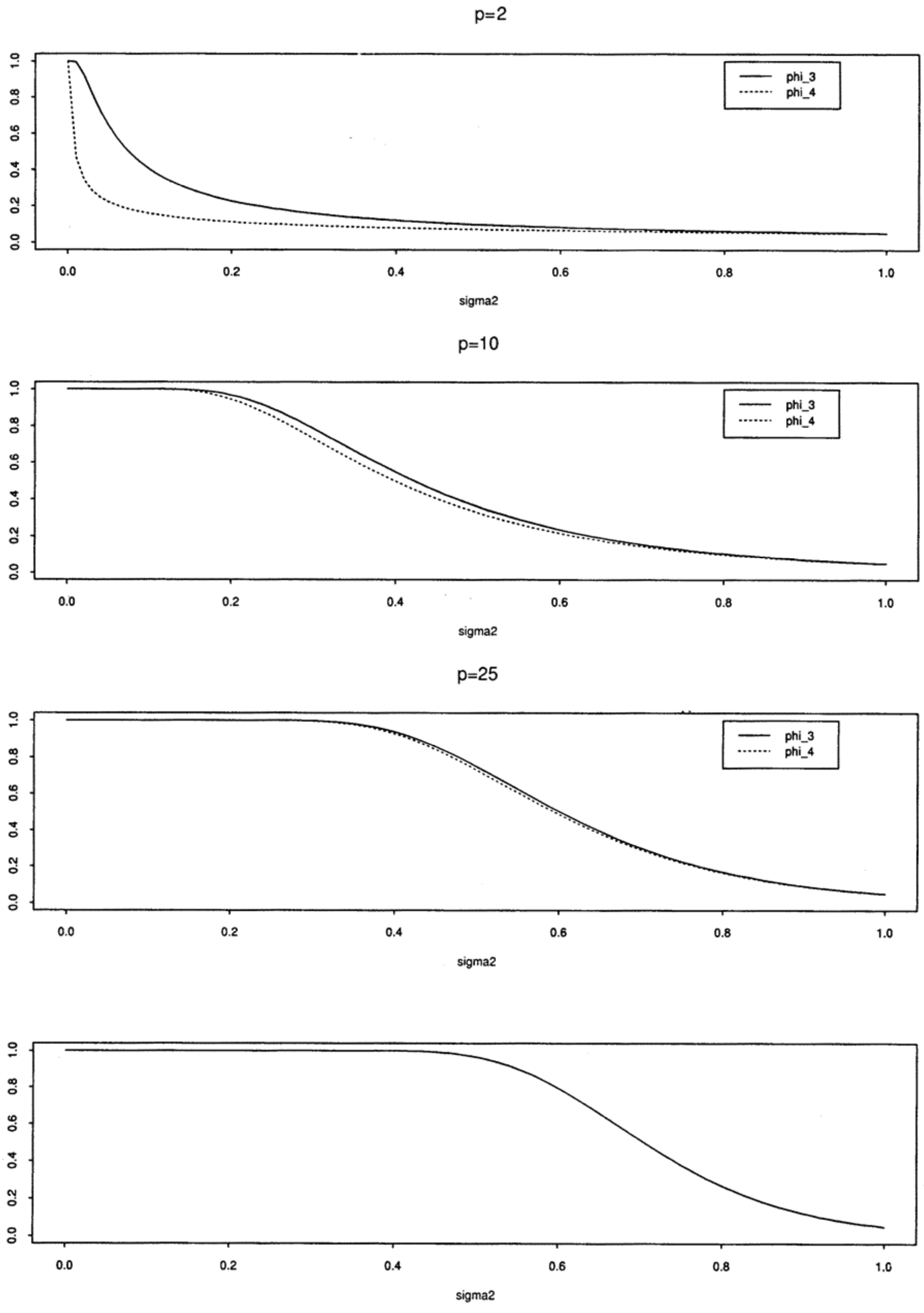

Figure 2. Power functions for the tests ϕ_3, and ϕ_4, where $\alpha = 0.05$, $\mu = 0$, $\sigma_0^2 = 1$, and $\rho = 0.5$, for p = 2, 10, 25, 50.

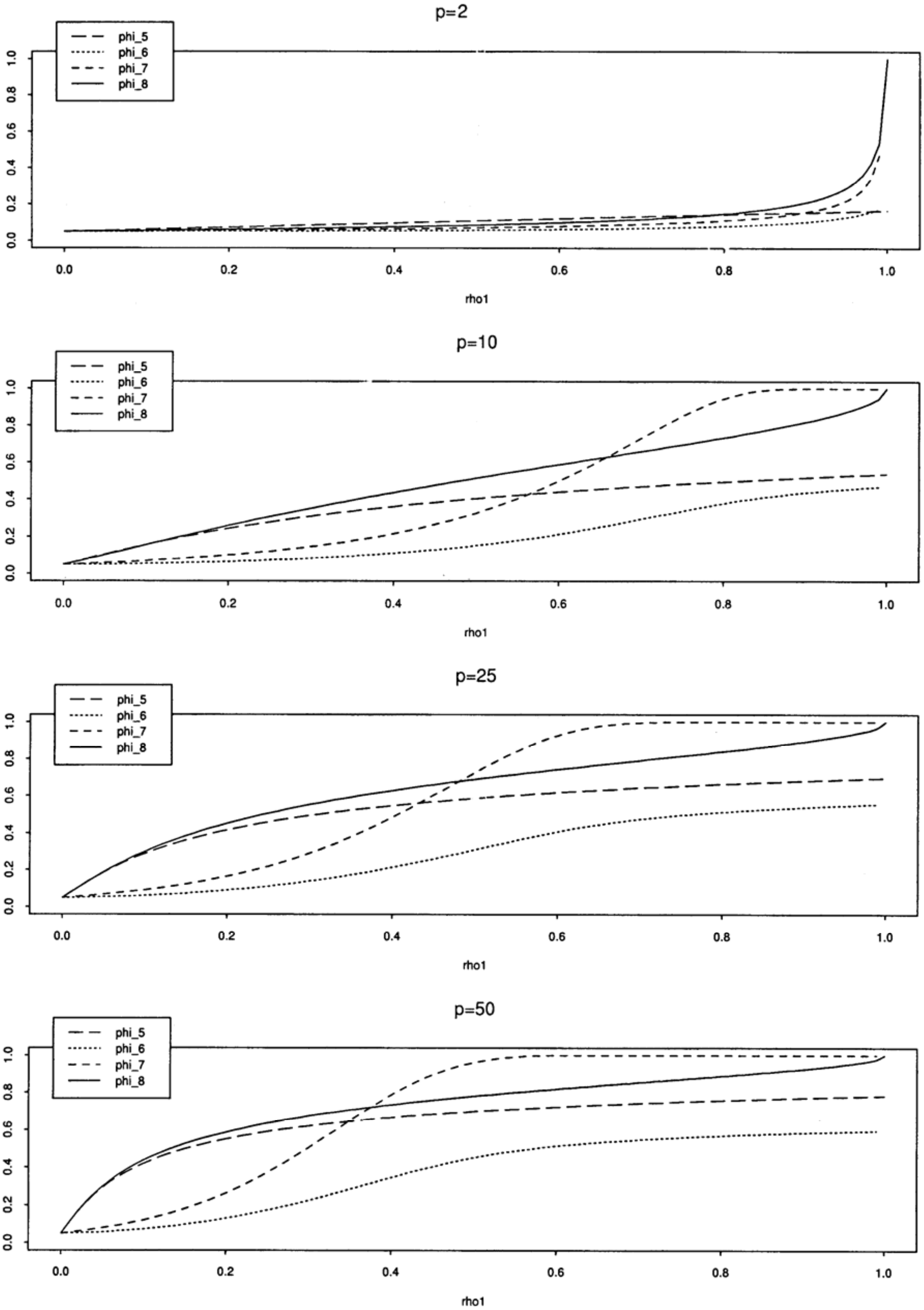

Figure 3. Power functions for the tests ϕ_5, ϕ_6, ϕ_7, and ϕ_8, where $\alpha = 0.05$, $\mu = 0$, and $\sigma^2 = 1$, for $p = 2, 10, 25, 50$.

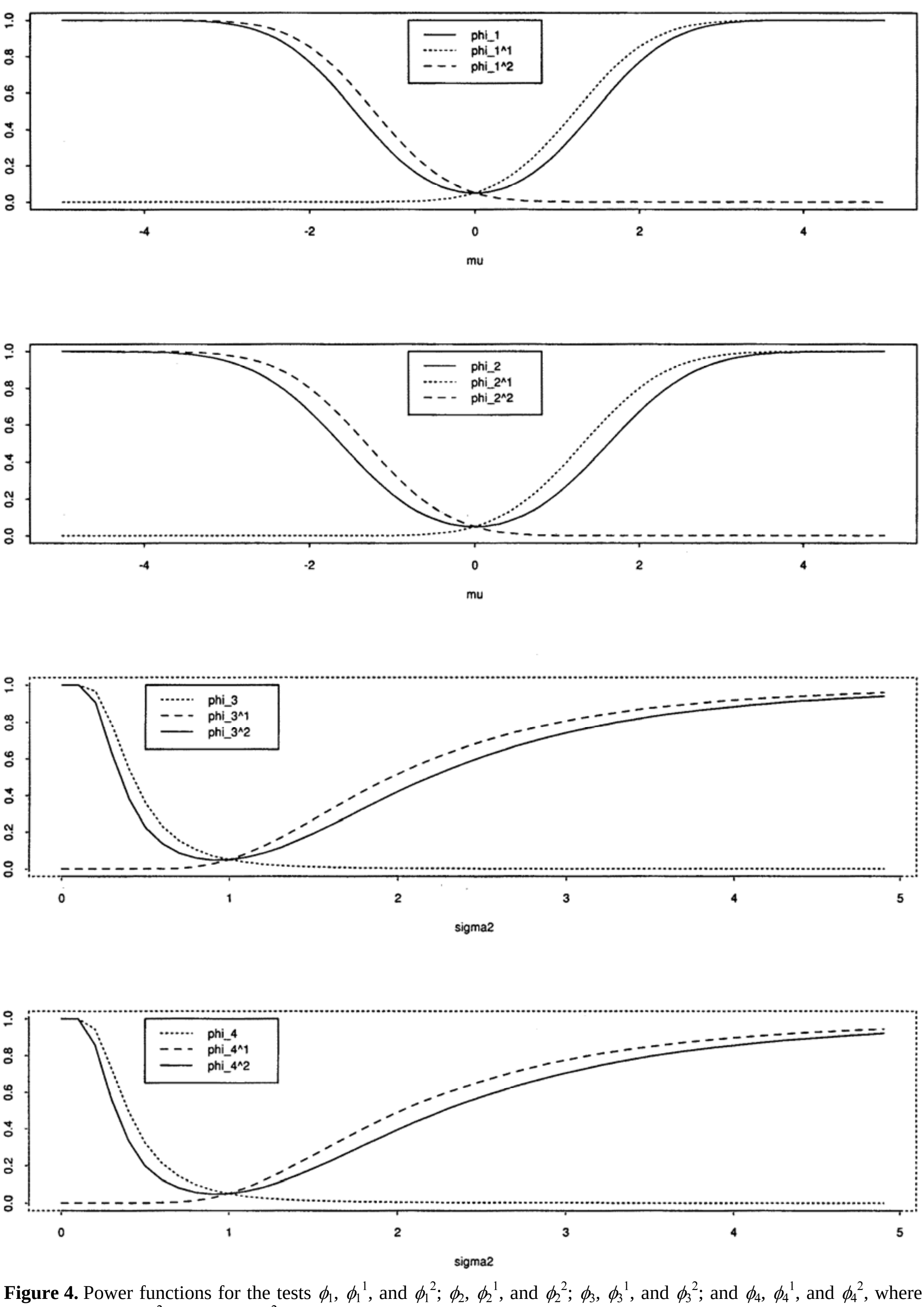

Figure 4. Power functions for the tests ϕ_1, $\phi_1^{\,1}$, and $\phi_1^{\,2}$; ϕ_2, $\phi_2^{\,1}$, and $\phi_2^{\,2}$; ϕ_3, $\phi_3^{\,1}$, and $\phi_3^{\,2}$; and ϕ_4, $\phi_4^{\,1}$, and $\phi_4^{\,2}$, where $\alpha = 0.05$, $\mu_0 = 0$, $\sigma_0^2 = 1$, $\mu = 0$, $\sigma^2 = 1$, $\rho = 0.5$, and $p = 10$.

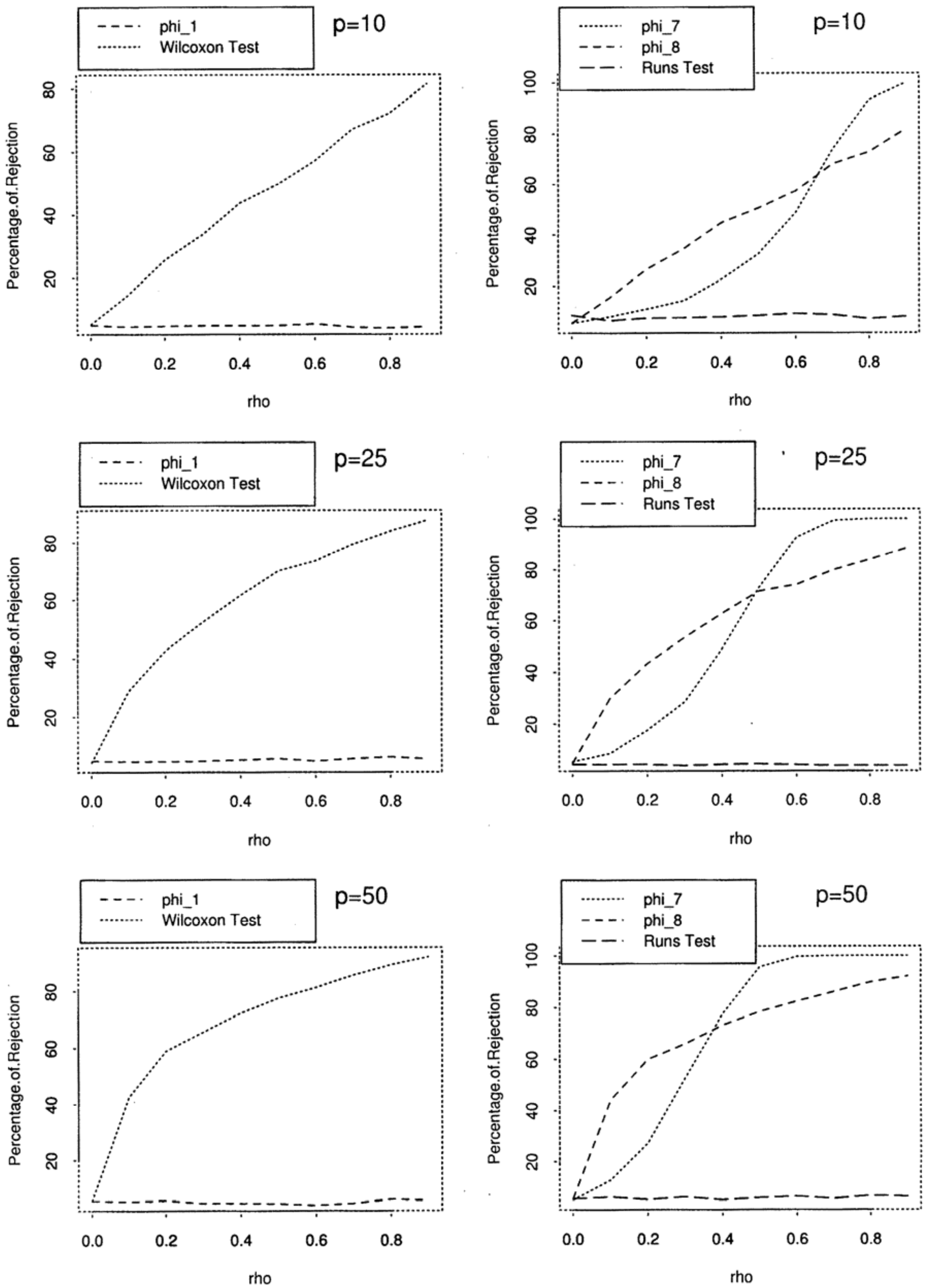

Figure 5. Percentage of rejecting H_0 for the test ϕ_1, and Wilcoxon test (left column); and percentage of rejecting H_0 for the tests ϕ_7, ϕ_8, and Runs test (right column), where $\alpha = 0.05$, $\mu_0 = 0$, $\sigma^2 = 1$, $p = 10, 25, 50$, for $\rho \in [0,1)$.

$\theta_1 = \rho / l$, $t_1 = \sum_{i<j} x_i x_j$, $\theta_2 = -(1+(p-2)\rho)/(2l)$, $t_2 = \sum_{i=1}^{p} x_i^2$, $\theta_3 = (1-\rho)\mu / l$, $t_3 = \sum_{i=1}^{p} x_i$, $l = \sigma^2(1+(p-1)\rho)(1-\rho)$, and $q_p(\theta_1, \theta_2, \theta_3)$ can be determined. Therefore the test function $\phi(t_1, t_2)$ given by $\phi(t_1,t_2,t_3) = \begin{cases} 1 & t_1 > c(t_2, t_3) \\ 0 & t_1 < c(t_2, t_3) \end{cases}$ is a UMPU test for testing $\begin{cases} H_0 : \rho = 0 \\ H_1 : \rho > 0 \end{cases}$ where $c(t_2, t_3)$ is so chosen that

$$P_{\theta_1=0}(T_1 > c(T_2,T_3) \mid T_2 = t_2, T_3 = t_3) = \alpha .$$

Note that $T_1 = (T_3^2 - T_2)/2$ and also the event $\{T_1 > c(T_2, T_3)\}$ depends on T_2 and T_3. Therefore, we have

$$P_{\theta_1=0}(T_1 > c(T_2,T_3) \mid T_2 = t_2, T_3 = t_3) = \begin{cases} 1 & t_1 > c(t_2, t_3) \\ 0 & t_1 < c(t_2, t_3) \end{cases},$$

which is equal to $\phi(t_1, t_2, t_3)$. If we use the method in the proof of Theorem 2.1, then we obtain a similar result.

Remark 2.2. If σ^2 is known but μ is unknown, we cannot have such an UMPU test for ρ by the method given in Theorem 2.1, because the density is not of the form (2.1).

Remark 2.3. As in the case $\rho = 0$, the test ϕ_1 is not UMP.

Remark 2.4. The tests ϕ_3^2, and ϕ_4^2 are not UMP or UMPU, because when $\rho = 0$ they are not UMP or UMPU (see Tate & Klett [20], and Parsian & Nematolahi [18]).

Remark 2.5. The tests ϕ_5 and ϕ_6 are not UMPU. To show this fact, compare these tests with ϕ_8 in Figure 3.

3. A Simulation Study

In this section we consider the effect of $\rho > 0$ on the test functions in Section 2. For this purpose we change ρ in the interval [0,1) and by simulation we study the robustness of these test functions. A good test function for μ or σ^2 should be robust when ρ changes in [0,1), but not for testing ρ.

For example, consider ϕ_1, the test function for testing $\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu \neq \mu_0 \end{cases}$, when $\mu_0 = 0$, and $\alpha = 0.05$. Figure 5 (left column) shows the percentage times of rejecting H_0 when $\mu = 0$ for $\rho \in [0,1)$. This test is robust, because percentage times of rejecting H_0 are approximately 5% for all $\rho \in [0,1)$. But, for example, the Wilcoxon test (nonparametric test for mean; see, e.g., Gibbons [13]) is not robust, i.e. percentage times of rejecting H_0 increases when ρ goes to 1.

However tests for ρ should not be robust, because they are sensitive to the change of ρ. Now consider ϕ_7, ϕ_8 and the Runs test (test of randomness; see, e.g., Gibbons [13]). Figure 5 (right column) shows the percentage times of rejecting H_0. At $\rho = 0$, the percentage times of rejecting H_0 for these tests are approximately 5%. When ρ increases the percentage goes up for ϕ_7 and ϕ_8, but not for the Runs test.

Remark 3.1. To simulate an $EN_p(\mu, \sigma^2, \rho)$, we use the Algorithm 8.1.2 of Tong [21] page 183, by an S-PLUS function. We generate 2000 times from an $EN_p(0, 1, \rho)$ for $\rho = 0, 0.1, \ldots, 0.9$. The complete result of this simulation can be downloaded from the author's homepage on the World Wide Web.

4. Applications

The main result of the previous section was the advantage of the following tests $\phi_1, \ldots, \phi_4$, ϕ_7 and ϕ_8. In this section, we try to answer the following question:

> Can we use the test functions $\phi_1, \ldots, \phi_4$, ϕ_7, and ϕ_8 in applied problems?

Suppose we have the assumption of normality. Consider the test functions ϕ_2, and ϕ_4. These test functions are useful if the parameter ρ is known, but in a real problem ρ is usually unknown and there is no estimate or nontrivial test for ρ. Therefore, we cannot test for μ or σ^2, unless ρ is known. Note that if μ or σ^2 is known then we can estimate ρ and there is a test for ρ (ϕ_7, or ϕ_8), and so the tests ϕ_1, and ϕ_3 are applied for testing μ, and σ^2, respectively after applying the test ϕ_7, or ϕ_8.

In the following, we point out some difficulties and restrictions for using tests ϕ_7, and ϕ_8.

Linear Models

The error terms in linear models usually have IID normal distribution with zero mean and unknown variance σ^2. One of the important problems in linear models is checking the assumption of IID or $\rho = 0$. Unfortunately, we cannot use test ϕ_8, because the sum of estimated errors is zero (see Arnold [1]).

Time Series

This case is similar to the case of linear models. However, in this case, the sum of estimated errors is not zero, so we can check the assumption of independence. Note that if we subtract the mean of observations from them (this transformation is usually used in time series, see, e.g., [2,3]) then the mean of estimated errors is near zero, so we cannot use test ϕ_8. Dufour and Roy [9] introduce some tests for checking independence assumption in exchangeable time series.

Statistical Quality Control

Suppose a process is generated by a system. If we cannot reject the assumption of normality then we can use the test functions ϕ_7, and ϕ_8, when variance or mean of the system is known, respectively. For an application of these tests see Section 7.2.1 of Leitnaker, Sanders and Hild [16].

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