SOME BOUNDARY VALUE PROBLEMS FOR A NON-LINEAR THIRD ORDER O.D.E.

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Abstract

Existence of periodic solutions for non-linear third order autonomous differential equation (O.D.E.) has not been investigated to as large an extent as non-linear second order. The popular Poincare-Bendixon theorem applicable to second order equation is not valid for third order equation (see [3]). This conclusion opens a way for further investigation.

Let us consider the following third order non-linear differential equation

$$x''' + ax' + f(x) = 0, \quad a > 0 \tag{1}$$

where *f(x)* is a continuous real-valued function with $xf(x) > 0$. Under this assumption, we shall establish the following theorem.

Theorem. Let us assume that there exist constants $D > 0$, $c_2 > 0$, α and β such that

(i) $a\alpha + f(x) \le 0 \le a\beta + f(x)$ for all $|x| \le D$

(ii) $c_2 > d$ where $d = \max(|\alpha|, |\beta|)$

(iii) $3M \le a^{3/2}(c_2 - d)$ with $M = \|f\|_\infty$ where $\|f\|_\infty = \max_{|x|\le D} |f(x)|$

(iv) $3m \le D$ where $m = \max(M, c_2 + d + \frac{6M}{a^{3/2}})$

Keywords: Autonomous differential equation; Banach space; Schauder's fixed theorem point; Boundary conditions

Then there exists at least one $\omega \in [\frac{\pi}{2\sqrt{a}}, \frac{3\pi}{2\sqrt{a}}]$ such that Equation (1) has a non-trivial solution satisfying the following boundary conditions:

$$x'(0) = x'(\omega), \qquad \int_0^\omega f(x(s))ds = 0 \tag{2}$$

Proof. For each $c_1 \in [\alpha, \beta]$, let us define the function $x(t) = x(t, c_1)$ as the solution of the following integral equation:

$$\begin{aligned} x(t) &= l + c_1 \sin\sqrt{a}t + c_2 \cos\sqrt{a}t + F(t, x(t)), \\ F(t, x(t)) &= -a^{-1}\int_0^t [1 - \cos\sqrt{a}(t-s)] f(x(s))ds \end{aligned} \tag{3}$$

Obviously,

$$x'(t) = \sqrt{a}\{c_1 \cos\sqrt{a}t - c_2 \sin\sqrt{a}t - a^{-1}\int_0^t f(x(s)) \sin\sqrt{a}(t-s)ds\}$$

One can easily verify that *x(t)* satisfies Equation (1). By (iii) for $|x| \le D$, we obtain

$$\left| a^{-1}\int_0^t f(x(s)) \sin\sqrt{a}(t-s)ds \right| \le \frac{3M}{a^{3/2}}.$$

Let $H(t) = x'(0) - x'(t)$, then clearly we have

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$$H(\frac{3\pi}{2\sqrt{a}}) = \sqrt{a}\left\{c_1 + c_2 - a^{-1}\int_0^{\frac{3\pi}{2\sqrt{a}}} f(x(s))\cos\sqrt{a}\, sds\right\},$$

and

$$H(\frac{\pi}{2\sqrt{a}}) = \sqrt{a}\left\{c_1 - c_2 + a^{-1}\int_0^{\frac{\pi}{2\sqrt{a}}} f(x(s))\cos\sqrt{a} sds\right\}.$$

For $|x| \le D$, we obtain the inequalities:

$$\left|\int_0^{\frac{3\pi}{2\sqrt{a}}} f(x(s))\cos\sqrt{a}\, sds\right| \le \frac{3M}{\sqrt{a}}$$

and

$$\left|\int_0^{\frac{\pi}{2\sqrt{a}}} f(x(s))\cos\sqrt{a} sds\right| \le \frac{M}{\sqrt{a}}.$$

Hence,

$$H(\frac{3\pi}{2\sqrt{a}}) \le \frac{3M}{a} + \sqrt{a}(c_1 - c_2) \le \frac{3M}{a} + \sqrt{a}(d - c_2) \le 0$$

and

$$H(\frac{\pi}{2\sqrt{a}}) \ge -\frac{M}{a} + \sqrt{a}(c_1 + c_2) =$$
$$\frac{-M + a^{3/2}(c_1 + c_2)}{a} \ge \frac{-3M + a^{3/2}(c_1 - c_2)}{a} \ge 0$$

Therefore, $H(\frac{3\pi}{2\sqrt{a}})H(\frac{\pi}{2\sqrt{a}}) \le 0$. If $H(\frac{3\pi}{2\sqrt{a}}) = 0$ or $H(\frac{\pi}{2\sqrt{a}}) = 0$ then $x'(0) = x'(\omega)$ with $\omega = \frac{3\pi}{2\sqrt{a}}$ or $\frac{\pi}{2\sqrt{a}}$. Hence, w.l.o.g, we may assume $H(\frac{3\pi}{2\sqrt{a}})H(\frac{\pi}{2\sqrt{a}}) < 0$ and the obvious continuity of *H(t)* implies the existence of one $\omega \in (\frac{\pi}{2\sqrt{a}}, \frac{3\pi}{2\sqrt{a}})$ such that $H(\omega)=0$ i.e. $x'(0) = x'(\omega)$. To complete the proof of the Theorem, we introduce the Banach space $B := C[0,\omega] \times \Re$. With norm $\|(\delta, l)\| := \|\delta\|_\infty + \|l\|$, $\|\delta\|_\infty := \max_{0\le t\le\omega}|\delta(t)|$. Now we define the map $T : B \to B$ by the rule: $T(\delta, l) = (\delta^*, l^*)$ where

$$\delta^* = l + c_1 \sin\sqrt{a}t + c_2\cos\sqrt{a}t + F(t, \delta(t))$$

and

$$l^* = l - \frac{1}{\omega}\int_0^\omega f(\delta^*(s))ds.$$

Then T is a continuous map from B into B. Next, we are going to prove that the closed subset K^* of B defined by

$$K^* := \{(\delta, l) \in B : |\delta| \le D, |l| \le 2m\}$$

is T-invariant. Since

$$|\delta^*| \le l + d + c_2 + \frac{6M}{a^{3/2}} \le 2m + m = 3m \le D,$$

it only remains to prove $|l^*| \le 2m$. For the proof of $|l^*| \le 2m$, three distinct cases will be considered.

Case I. $|l| \le m$: Since $|\delta^*| \le D$, it follows that

$$\left|\frac{1}{\omega}\int_0^\omega f(\delta^*(s))ds\right| \le M \le m$$

and hence,

$$-2m \le l - \frac{1}{\omega}\int_0^\omega f(\delta^*(s))ds \le 2m \quad \text{i.e.} \quad |l^*| \le 2\text{m}. \quad (4)$$

Case II. $m < l < 2m$: Obviously

$$|\delta^* - l| \le d + c_2 + \frac{6M}{a^{3/2}} \le m,$$

which implies that $3m \ge \delta^* \ge l - m > 0$ and hence, $f(\delta^*) > 0$, since by our assumption $\delta^* f(\delta^*) > 0$. Therefore, we obtain

$$0 < \frac{1}{\omega}\int_0^\omega f(\delta^*)ds \le M \le m,$$

which clearly means:

$$0 < l - m \le l^* \le l \le 2m \quad \text{or} \quad |l^*| \le 2\text{m} \quad (5)$$

Case III. $-2m \le l < -m$: With similar arguments we obtain $-3m < \delta^* < 0$ implying

$$0 < -\frac{1}{\omega}\int_0^\omega f(\delta^*)ds < m$$

and hence,

$$-2m \le l < l^* = l - \frac{1}{\omega}\int_0^\omega f(\delta^*(s))ds < l + m < 0 \quad \text{i.e.} \quad |l^*| \le 2\text{m} \quad (6)$$

Thus, K^* is a *T*-invariant closed subset of the Banach space *B*. Using Schauder's fixed point Theorem (for more discussion on the subject see [1,4], there exists at least one element $(\delta,\gamma)\in K^*$ such that $T(\delta,\gamma)=(\delta,\gamma)$ i.e.

$$\delta(t)=\delta(t;c_1)=\gamma+c_1\sin\sqrt{a}\,t+c_2\cos\sqrt{a}\,t+F(t,\delta(t;c_1))$$

and

$$\int_0^{\omega}(\delta(s;c_1))ds=0\,,$$

which completes the proof of the Theorem.

Example. Let us consider the equation $x'''+x'+\sin x=0$. With taking $\alpha=-1, \beta=-1, d=1, c_2=4, D=34, m=11, M=1$, all assumptions of the Theorem are fulfilled. Hence, there exists $\omega\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ such that $x'(0)=x'(\omega)$ and $\int_0^{\omega} f\sin(x(s))ds=0$.

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