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# Bracket Products on Locally Compact Abelian Groups

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## Abstract

We define a new function-valued inner product on $L^2$(G), called $\varphi$-bracket product, where $G$ is a locally compact abelian group and $\varphi$ is a topological isomorphism on $G$. We investigate the notion of $\varphi$-orthogonality, Bessel's Inequality and $\varphi$-orthonormal bases with respect to this inner product on $L^2$(G).

**Keywords:** Bracket product; Locally compact abelian group; $\varphi$-Orthonormal basis; Bessel's Inequality

### Introduction and Preliminaries

The bracket product in $L^2(\mathbb{R}^n)$, which was originally introduced in [1] and extended in [10], play an important role in the analysis of shift-invariant spaces. For $f, g \in L^2(\mathbb{R}^n)$, the bracket product is defined by $[f, g] := \sum_{\alpha \in Z^n} f\overline{g}(. + \alpha)$. Then $[f, g]$ is a well defined element in $L^1(\mathbb{T}^n)$ and satisfies $\|[f, f]\|_{L^1(\mathbb{T}^n)} = \|f\|^2_{L^2(\mathbb{R}^n)}$. Casazza and Lammers have extended this notion and analysed the properties of the bracket product of two $L^2(\mathbb{R})$-functions $f, g$ defined for $a \in \mathbb{R}^+$ by $< f, g >_a = \sum_{n \in Z} f\overline{g}(. - na)$. They have then applied this to show that all operators associated to a Gabor system have a compression with regard to this bracket product. Moreover, in [5**Error! Reference source not found.**] the authors have made use of an extension of this notion to $\mathbb{R}^n$ for their results on reproducing systems.

In this paper we define a new inner product on $L^2$(G) called $\varphi$-bracket product, where $G$ is a locally compact abelian (LCA) group and $\varphi$ is a topological isomorphism on $G$. We investigate its properties and we show that this bracket product gives rise to several results similar to the ones in the standard Hilbert space theory. This theory, originated from David Hilbert's work (1862-1943), especially $L^2$-spaces, is essential in the development of wavelet transform analysis. One of the nicest features of these spaces is that their geometry is very much similar to the familiar Euclidian geometry.

The $\varphi$-bracket product is useful in two ways. First it is a unified approach to all the bracket products mentioned above, on $\mathbb{R}^n$. Secondly, it is applicable to extend many ideas and constructions from the theory of shift-invariant spaces, factorable operators and Weyl-Heisenberg frames on $\mathbb{R}^n$, to the setting of LCA groups in a more general and different way. In a forthcoming paper we will work out these various applications on LCA groups (see also [7]).

The rest of this paper is organized as follows. Section 2 sets out the definition and elementary properties of the $\varphi$-bracket product on a LCA group. In section 3 we study $\varphi$-orthonormal sequences and $\varphi$-bases in $L^2$(G) where $G$ is a second countable LCA group. We show that there is a close relationship between $\varphi$-orthogonality and usual orthogonality in $L^2$(G). Moreover, we prove Bessel's Inequality for $\varphi$-orthonormal sequences and Parseval Identity for $\varphi$-

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bases. Finally we show that $L^2(G)$ admits such bases just as in the standard Hilbert space theory.

Before embarking on our study, we recall some basic facts from the theory of LCA groups. For general references on this theory, we refer to [3,6].

Let $G$ be a locally compact abelian group. It is well known that such a group possesses a Haar measure that is unique up to a multiplication by constants. Suppose $H$ is a closed subgroup of $G$. Let $G/H$ be the quotient group whose Haar measure is $\mu$ (which is unique up to a constant factor). If this factor is suitably chosen we have

$$\int_G f(x)dx = \int_{G/H}\int_H f(xy)dy\,d\mu(xH) \quad f \in L^1(G).$$

This identity is known as Weil's formula.

We shall denote the dual group of $G$ by $\hat{G}$. Let the Fourier transform $\hat{}: L^1(G) \to C_0(\hat{G})$, $f \to \hat{f}$, be defined by $\hat{f}(\xi) = \int_G f(x)\overline{\xi}(x)dx$. The Fourier transform can be extended to a unitary isomorphism from $L^2(G)$ to $L^2(\hat{G})$ known as the Plancherel transform [3].

A subgroup $L$ of $G$ is called a uniform lattice if it is discrete and co-compact (i.e $G/L$ is compact). The subgroup $L^\perp = \{\xi \in \hat{G}; \xi(L) = \{1\}\}$ is called the annihilator of $L$ in $\hat{G}$ which is a uniform lattice in $\hat{G}$ (for more details see [8]).

## Definition and Elementary Properties

Throughout this paper we always assume that $G$ is a second countable LCA group. Suppose $L$ is a uniform lattice in $G$, and $\varphi: G \to G$ is a topological isomorphism.

If $f, g \in L^2(G)$, then $f\overline{g} \in L^1(G)$ and by Weil's formula we have

$$\begin{aligned}
&\int_{G/\varphi(L)} \sum_{k\in L} |f\overline{g}(x\varphi(k^{-1}))|\,d\dot{x} \\
&= \int_{G/\varphi(L)} \sum_{\varphi(k)\in\varphi(L)} |f\overline{g}(x\varphi(k^{-1}))|\,d\dot{x} \\
&= \int_G |f\overline{g}(x)|\,dx \\
&= \|f\overline{g}\|_1 < \infty,
\end{aligned}$$

where $\dot{x} = x\varphi(L)$, and $d\dot{x}$ is the Haar measure on $G/\varphi(L)$. Thus for almost all $x \in G$, $\sum_{k\in L} f\overline{g}(x\varphi(k^{-1}))$ converges. Now we can define the $\varphi$-bracket product of any $f, g \in L^2(G)$ as a function-valued inner product on $L^2(G)$.

**Definition 1.** *Let $f, g \in L^2(G)$. The $\varphi$-bracket product of $f, g$ is defined by $[f,g]_\varphi(x) = \sum_{k\in L} f\overline{g}(x\varphi(k^{-1}))$, for all $x \in G$. We define the $\phi$-norm of $f$ as $\|f\|_\varphi(x) = ([f,f]_\varphi(x))^{1/2}$.*

Note that $[f,g]_\varphi$ is $\varphi$-periodic, i.e. for every $l \in L, x \in G$, $[f,g]_\varphi(x\varphi(l)) = [f,g]_\varphi(x)$. In other words $[f,g]_\varphi$ is constant on $\varphi(L)$-cosets. So one may consider the $\varphi$-bracket product of $f, g \in L^2(G)$, as the mapping $[.,.]_\varphi : L^2(G) \times L^2(G) \to L^1(G/\varphi(L))$, defined by $[f,g]_\varphi(\dot{x}) = \sum_{k\in L} f\overline{g}(x\varphi(k^{-1}))$ for all $\dot{x} \in G/\varphi(L)$. Throughout, we will use both of the above notations interchangeably.

**Example 2.** *Consider $G = \mathbb{R}, L = \mathbb{Z}$ in the above definition. Fix $a \in \mathbb{R}^+$. Then $\varphi: \mathbb{R} \to \mathbb{R}$, given by $\varphi(x) = ax$ is a topological isomorphism and the mapping $[.,.]_\varphi : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^1[0,a]$, defined by $[f,g]_\varphi(x) = \sum_{n\in\mathbb{Z}} f\overline{g}(x - na)$ is the $a$-pointwise inner product of $f$ and $g$ introduced by Casazza and Lammers in [2]. If $\varphi$ is the identity function on $\mathbb{R}$ then the $\varphi$-bracket product is exactly the bracket product defined by Ron and Shen [10].*

**Example 3.** *Let $G = \mathbb{R}^n \times \mathbb{Z}^n \times \mathbb{T}^n \times Z_n$, for $n \in \mathbb{N}$, where $Z_n$ is the finite abelian group $\{1,2,...,n\}$ of residues modulo $n$. Then $L = \mathbb{Z}^n \times \mathbb{Z}^n \times \{1\} \times Z_n$ is a uniform lattice in $G$. Let $A$ be an invertible $n \times n$ real matrix and fix $l \in \mathbb{Z}^n$. Define $\varphi: G \to G$ by $\varphi(x,m,t,p) = (Ax, l+m, t, p)$, for every $x \in \mathbb{R}^n$, $m \in \mathbb{Z}^n$, $t \in \mathbb{T}^n$, $p \in Z_n$. The $\phi$-bracket product is given by $[f,g]_\varphi(x,m,t,p) = \sum_{k\in\mathbb{Z}^n, n\in\mathbb{Z}^n, q\in Z_n} f\overline{g}(x - Ak, m - l + n, t-1, p-q)$.*

The $\varphi$-bracket product behaves in some ways like the usual inner product on a Hilbert space. The following properties are immediate from the definition.

**Proposition 4.** *Let $f, g \in L^2(G)$ and $c \in \mathbb{C}$. Then the following properties hold:*

- (1) $\int_{G/\varphi(L)} [f,g]_\varphi(\dot{x})d\dot{x} = \langle f, g\rangle_{L^2(G)}$.

• (2) $[f+g,h]_\varphi=[f,h]_\varphi+[g,h]_\varphi$, $[cf,g]_\varphi=c[f,g]_\varphi$.

• (3) $[f,f]_\varphi\ \geq 0$ and $[f,g]_\varphi=\overline{[g,f]}_\varphi$.

• (4) $[f,f]_\varphi=0$ if and only if $f=0$.

• (5) $|[f,g]_\varphi|\leq\|f\|_\varphi\|g\|_\varphi$ (Cauchy Schwartz Inequality).

• (6) $\|f+g\|_\varphi\leq\|f\|_\varphi+\|g\|_\varphi$ (Triangle Inequality).

• (7) $\|f+g\|_\varphi^2+\|f-g\|_\varphi^2=2(\|f\|_\varphi^2+\|g\|_\varphi^2)$ (The Parallelogram Law).

• (8) $[f,g]_\varphi=1/4[\|f+g\|_\varphi^2-\|f-g\|_\varphi^2+i(\|f+ig\|_\varphi^2-\|f-ig\|_\varphi^2)]$ (The Polarization Identity).

(Notice that ((1)-(8) hold almost every where).

It is easy to see that The Pythagorean Theorem also holds for $\varphi$-bracket products, i.e. for $f_1,...,f_n\in L^2(G)$ if $[f_i,f_j]_\varphi=0$ for $i\neq j$, then $\left\|\sum_{i=1}^n f_i\right\|_\varphi^2=\sum_{i=1}^n\|f_i\|_\varphi^2$.

Similar to the usual inner product, we may define $\varphi$-orthogonality.

**Definition 5.** *Let $f,g\in L^2(G)$. We say $f$ and $g$ are $\varphi$-orthogonal if $[f,g]_\varphi=0$. A sequence $(g_n)_{n\in\mathbb{N}}\subseteq L^2(G)$ is called $\varphi$-orthonormal if $[g_n,g_m]_\varphi=0$, for all $n\neq m\in\mathbb{N}$ and $\|g_n\|_\varphi=1$ for all $n\in\mathbb{N}$. For $E\subseteq L^2(G)$, the $\varphi$-orthogonal complement of $E$ is $E^{\perp\varphi}=\{g\in L^2(G);\ [f,g]_\varphi=\ 0\,a.e.$, for all $f\in E\}$.*

For any subset $E\subseteq L^2(G)$, there is a relation between $\varphi$-orthogonal complement of $E$ and its orthogonal complement in $L^2$(G) which is given in the next proposition. First we state a lemma whose proof is easy and so omitted.

**Lemma 6.** *Let $f,g\in L^2(G)$. Denote by $B_\varphi(G)$ the set of all $\varphi$-periodic functions in $L^\infty(G)$, i.e. $B_\varphi(G)=\{h\in L^\infty(G), h(x\varphi(k))=h(x), forall\ k\in L\}$. Then for all $h\in B_\varphi(G)$, $[fh,g]_\varphi=h[f,g]_\varphi$ and $[f,\overline{h}g]_\varphi=h[f,g]_\varphi$.*

Now we can prove the following proposition.

**Proposition 7.** *Let $E\subseteq L^2(G)$, and $B_\varphi(G)$ be as in Lemma 6. Then $E^{\perp\varphi}=\bigcap_{h\in B_\varphi(G)}(hE)^\perp$.*

*Proof.* Let $f\in E^{\perp\varphi}$. Then for $h\in B_\varphi(G)$ and $g\in E$, by Proposition 4 (1) and Lemma 6 we have $\langle f,hg\rangle_{L^2(G)}=\int_{G/\varphi(L)}[f,hg]_\varphi(\dot{x})d\dot{x}=\int_{G/\varphi(L)}\overline{h}(\dot{x})[f,g]_\varphi(\dot{x})d\dot{x}=0$. So, $f\in\bigcap_{h\in B_\varphi(G)}(hE)^\perp$.

Now let $f\in\bigcap_{h\in B_\varphi(G)}(hE)^\perp$ and $g\in E$. For $n\in\mathbb{N}$, define $h_n(\dot{x})=[f,g]_\varphi(\dot{x})$, when $|[f,g]_\varphi(\dot{x})|\leq n$, and $h_n(\dot{x})=0$ otherwise. Then obviously $h_n\in B_\varphi(G)$ and we have $0=\langle f,h_ng\rangle_{L^2(G)}=\int_{G/\varphi(L)}[f,h_ng]_\varphi(\dot{x})d\dot{x}=\int_{G/\varphi(L)}\overline{h_n(\dot{x})}[f,g]_\varphi(\dot{x})d\dot{x}=\int_{G/\varphi(L)}|h_n(\dot{x})|^2\,d\dot{x}$. Hence $[f,g]_\varphi=0\,a.e.$, that is $f\in E^{\perp\varphi}$.

We will develop more, the notion of $\varphi$-orthogonality in the next section.

## Bessel's Inequality and $\varphi$-Orthonormal Bases

This section is devoted to establishing Bessel's Inequality for $\varphi$-orthonormal sequences in $L^2$(G). We will also define a $\varphi$-orthonormal basis in terms of an orthonormal basis in $L^2$(G) and prove Parseval's Identity in this case.

First we would like to consider $\varphi$-orthogonality in terms of orthogonality in $L^2$(G). The key is the following proposition.

For $\gamma\in\hat{G}$, denote by $M_\gamma$ the modulation operator on $L^2$(G), i.e. $M_\gamma f(x)=\gamma(x)f(x)$, for all $f\in L^2(G)$.

**Proposition 8.** *Suppose $f,g\in L^2(G)$, and $\gamma\in\varphi(L)^\perp$, where $\varphi(L)^\perp$ is the annihilator of $\varphi(L)$ in $\hat{G}$. Then $\widehat{[f,g]_\varphi}(\gamma)=\langle f,M_\gamma g\rangle_{L^2(G)}$.*

*Proof.* Since $\gamma(\varphi(k^{-1}))=0$ for all $k\in L$, for $f,g\in L^2(G)$ we have

$$\widehat{[f,g]_\varphi}(\gamma) = \int_{G/\varphi(L)} [f,g]_\varphi(\dot{x})\overline{\gamma}(\dot{x})d\dot{x}$$
$$= \int_{G/\varphi(L)} \sum_{k\in L} f\overline{g}(x\varphi(k^{-1}))\overline{\gamma}(x\varphi(k^{-1}))d\dot{x}$$
$$= \int_{G/\varphi(L)} \sum_{k\in L} f(x\varphi(k^{-1}))\overline{M_\gamma g}(x\varphi(k^{-1}))d\dot{x}$$
$$= \int_{G/\varphi(L)} [f, M_\gamma g](\dot{x})d\dot{x}$$
$$= \langle f, M_\gamma g\rangle_{L^2(G)}.$$

We obtain immediately

**Corollary 9.** *Let $f,g \in L^2(G)$. Then $f,g$ are $\varphi$-orthogonal if and only if $\overline{span}\{M_\gamma f;\gamma\in\varphi(L)^\perp\}$ and $\overline{span}\{M_\gamma g;\gamma\in\varphi(L)^\perp\}$ are bi-orthogonal in $L^2(G)$.*

Whence we can characterize $\varphi$-orthonormal systems $(g_n)_{n\in\mathbb{N}}$ in $L^2(G)$.

**Theorem 10.** *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(G)$. Then $(g_n)_{n\in\mathbb{N}}$ is $\varphi$-orthonormal if and only if $(M_\gamma g_n)_{n\in\mathbb{N},\gamma\in\varphi(L)^\perp}$ is orthonormal.*

*Proof.* Suppose $(g_n)_{n\in\mathbb{N}}$ is a $\varphi$-orthonormal system. Let $\gamma,\eta\in\varphi(L)^\perp$. For $n\neq m$, $\gamma\neq\eta$ we have $\langle M_\gamma g_n, M_\eta g_m\rangle = 0$, by Corollary 9. For $n=m$, first note that by [6], $G/\varphi(L)$ is topologically isomorphic to $G/L$ and so it is compact. Thus by [3], $\varphi(L)^\perp(=\widehat{G/\varphi(L)})$ is an orthonorma basis for $L^2(G/\varphi(L))$. Hence we have

$$\langle M_\gamma g_n, M_\eta g_n\rangle$$
$$= \int_G M_\gamma g_n(x)\overline{M_\eta g_n(x)}dx$$
$$= \int_G |g_n(x)|^2\gamma(x)\overline{\eta}(x)dx$$
$$= \int_{G/\varphi(L)} \sum_{\varphi(k)\in\varphi(L)} |g_n(x\varphi(k^{-1}))|^2\gamma\overline{\eta}(x\varphi(k^{-1}))d\dot{x}$$
$$= \int_{G/\varphi(L)} \|g_n\|_\varphi^2(\dot{x})\gamma(\dot{x})\overline{\eta}(\dot{x})d\dot{x}$$
$$= \int_{G/\varphi(L)} \gamma(\dot{x})\overline{\eta}(\dot{x})d\dot{x}$$
$$= \delta_{\gamma,\eta},$$

which completes the proof. The proof of the converse is similar.

In the following theorem we prove Bessel's Inequality for $\varphi$-orthonormal sequences in $L^2(G)$.

**Theorem 11.** *(Bessel's Inequality) Let $f\in L^2(G)$ and $(g_n)_{n\in\mathbb{N}}$ be a $\varphi$-orthonormal sequence in $L^2(G)$. Then*

$$\sum_{n\in\mathbb{N}} |[f,g_n]_\varphi(\dot{x})|^2 \le \|f\|_\varphi^2(\dot{x}), for\ a.e.\ x\in G. \quad (1)$$

*Proof.* Observe that $[f,g_n]_\varphi g_n \in L^2(G)$ (for a more general statement of this fact see Remark 13). Indeed, by Weil's formula and Cauchy Schwartz Inequality for $\varphi$-bracket products (Proposition 4 (5)), we have

$$\|[f,g_n]_\varphi g_n\|_{L^2(G)}^2$$
$$= \int_{G/\varphi(L)} \sum_{\varphi(l)\in\varphi(L)} |[f,g_n]_\varphi g_n(x\varphi(l^{-1}))|^2 d\dot{x}$$
$$= \int_{G/\varphi(L)} |[f,g_n]_\varphi(\dot{x})|^2 \sum_{\varphi(l)\in\varphi(L)} |g_n(x\varphi(l^{-1}))|^2 d\dot{x}$$
$$= \int_{G/\varphi(L)} |[f,g_n]_\varphi(\dot{x})|^2 \|g_n\|_\varphi^2(\dot{x})d\dot{x}$$
$$\le \int_{G/\varphi(L)} \|f\|_\varphi^2(\dot{x})d\dot{x}$$
$$= \|f\|_2^2 < \infty,$$

Where in the second equality we have used $\varphi$-periodicity of $[f,g_n]_\varphi$ and the inequality is followed by Cauchy Schwartz Inequality for $\varphi$-bracket products together with orthonormality of $(g_n)_{n\in\mathbb{N}}$. Now fix $m\in\mathbb{N}$. Then $0\le [f-\sum_{i=1}^m [f,g_i]_\varphi g_i, f-\sum_{i=1}^n [f,g_i]_\varphi g_i]_\varphi = [f,f]_\varphi - \sum_{i=1}^m |[f,g_i]_\varphi|^2$. Since $m$ was arbitrary we conclude (1).

**Remark 12.** *If $f\in L^2(G)$ and $(g_n)_{n\in\mathbb{N}}$ is a $\varphi$-orthonormal sequence in $L^2(G)$ then $\sum_{n\in\mathbb{N}}[f,g_n]_\varphi g_n$ converges in $L^2(G)$. Indeed, we have*

$$\left\|\sum_{i=1}^m [f,g_i]_\varphi g_i\right\|_{L^2(G)}^2$$
$$= \int_{G/\varphi(L)} \left\|\sum_{i=1}^m [f,g_i]_\varphi g_i\right\|_\varphi^2(\dot{x})d\dot{x}$$
$$= \int_{G/\varphi(L)} \sum_{i=1}^m |[f,g_i]_\varphi(\dot{x})|^2 d\dot{x}.$$

By Bessel's Inequality and Monotone Convergence

Theorem, $\sum_{n\in\mathbb{N}} |[f, g_n]_\varphi(\dot{x})|^2$ converges in $L^1(G/\varphi(L))$. So if $m\to\infty$ then the right hand side of the above equality tends to zero. Thus $\sum_{n\in\mathbb{N}}[f, g_n]_\varphi g_n$ converges in $L^2(G)$.

**Remark 13.** *Notice that for $f, g\in L^2(G)$ the function $[f, g]_\varphi g$ need not generally be in $L^2(G)$. In fact, put $f(x)=g(x)=\chi_{[0,a]}x^{-1/3}$ in Example 2. Then $f, g\in L^2(\mathbb{R})$, but $[f, g]_\varphi g$ is not in $L^2(\mathbb{R})$.*

We Say $g\in L^2(G)$ is $\varphi$-bounded if there exists $M>0$ so that $\|g\|_\varphi \le M$ $a.e.$ Observe that if $f, g, h\in L^2(G)$ and $g, h$ are $\varphi$-bounded then $[f, g]_\varphi h\in L^2(G)$.

Now we are ready to define a $\varphi$-orthonormal basis in terms of an orthonormal basis in $L^2(G)$. To this end, we establish the following theorem.

**Theorem 14.** *If $(g_n)_{n\in\mathbb{N}}$ is a $\varphi$-orthonormal sequence in $L^2(G)$, the following are equivalent.*

- (1) $(g_n)_{n\in\mathbb{N}}$ is a maximal $\varphi$-orthonormal sequence, i.e. $(g_n)_{n\in\mathbb{N}}$ is not contained in any other $\varphi$-orthonormal set.
- (2) If $[f, g_n]_\varphi = 0$ a.e. for all $n\in\mathbb{N}$, then $f=0$ a.e. (completeness).
- (3) For each $f\in L^2(G), f=\sum_{n\in\mathbb{N}}[f, g_n]_\varphi g_n$ a.e.
- (4) $\|f\|_\varphi^2 = \sum_{n\in\mathbb{N}} |[f, g_n]_\varphi|^2$ a.e. for all $f\in L^2(G)$ (Parseval Identity).
- (5) $[f, g]_\varphi = \sum_{n\in\mathbb{N}}[f, g_n]_\varphi [g_n, g]_\varphi$ $a.e.$
- (6) $\{M_\gamma g_n\}_{n\in\mathbb{N},\gamma\in\varphi(L)^\perp}$ is an orthonormal basis for $L^2(G)$.

*Proof.* It is not difficult to mimic the standard proofs for a usual orthonormal sequence in a Hilbert space to obtain the implications $(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(4)\Rightarrow(5)$ $\Rightarrow(1)$. We show equivalence of conditions (2) and (6) and this will complete the proof. Assume (2). Since $(g_n)_{n\in\mathbb{N}}$ is a $\varphi$-orthonormal sequence, $\{M_\gamma g_n\}_{n\in\mathbb{N},\gamma\in\varphi(L)^\perp}$ is an orthonormal system, by Theorem 10. So by [4], it is enough to show that it is complete. Let $f\in L^2(G)$. If $<f, M_\gamma g_n>=0$ for all $n\in\mathbb{N}$, $\gamma\in\varphi(L)^\perp$ then $\widehat{[f, g_n]_\varphi}(\gamma)=0$ for every $n\in\mathbb{N}$, $\gamma\in\varphi(L)^\perp$, which implies $[f, g_n]_\varphi = 0$ for all $n\in\mathbb{N}$. Thus by (2), $f$ = 0 a.e. Similarly (6) implies (2).

We call a $\varphi$-orthonormal sequence having one of the properties (1)-(6) in Theorem 14, a $\varphi$-orthonormal basis.

Now it is natural to ask if $\varphi$-orthonormal bases always exist in $L^2(G)$. The following proposition gives an affirmative answer to this question.

**Proposition 14.** $L^2(G)$ *admits a* $\varphi$*-orthonormal basis.*

*Proof.* By Zorn's Lemma the collection of $\varphi$-orthonormal sets ordered by inclusion has a maximal element and maximality is equivalent to property (1) in Theorem 14.

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