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ω-Independent and Dissociate Sets on Compact Commutative Strong Hypergroups

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Abstract

In this paper we define ω-independent (a weak-version of independence), Kronecker and dissociate sets on hypergroups and study their properties and relationships among them and some other thin sets such as independent and Sidon sets. These sets have the lacunarity or thinness property and are very useful indeed. For example Varopoulos used the Kronecker sets to prove the Malliavin theorem. In the final section we bring some examples and find ω-independent and dissociate sets on a compact countable hypergroup of Dunkle and Ramirez, the dual Chebychev polynomial hypergroup, and some other polynomial hypergroups from Lasser.

Keywords: Hypergroups; ω-independence; Kronecker sets; Dissociate sets; Riesz products

1. Introduction and Notation

Hypergroups were introduced by Dunkl [2], Jewett [10], and Spector [17] in early 70's, although some similar structures had already been studied (see [15], [1]). Since then some aspects of harmonic analysis on groups have been generalized to hypergroups. Our aim in this paper is to study some special subsets of hypergroups and dual hypergroups that have lacunarity or thinness property and have many applications in harmonic analysis and related topics. In a series of papers [19-21], Vrem has studied lacunarity on hypergroups. Also C. Finet has extended some results about Riesz, Nicely placed, Shapiro and Rosenthal sets to the duals of compact hypergroups [4]. In this paper, we introduce the notion of $\omega-$independent sets on hypergroups which is the extension of both of independent sets on hypergroups [21] and independence property on groups. For a group G, a subset S of G is called independent if for every distinct elements $x_1,...,x_n$ of S and every integers $k_1,...,k_n$, the equation $x_1^{k_1}...x_n^{k_n}=e$ implies $x_1^{k_1}=...=x_n^{k_n}=e$ [9].

For a compact commutative hypergroup K, a subset E of Γ, the dual of K, is called independent by Vrem [21], if $1\notin E$ and for any finite subset $F=\{\xi_1,...,\xi_n\}$ of E:

(i) for each $\xi\in E\setminus F$, $\langle F\rangle\cap\langle\xi\rangle=\{1\}$; and

(ii) for all $\psi_i\in\langle\xi_i\rangle$ $(i=1,...,n)$, $\{\psi_1\}*...*\{\psi_n\}$ is a singleton in Γ.

Although such an independent subset of Γ does not necessarily belong to the center of hypergroup and in the case of Γ being a group this definition coincides

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with the definition of independent subsets of groups, if Γ is only a hypergroup the property (ii) of the above definition is quite restricted. By an $\omega-$independent we mean an independent set which is not confined by (ii) (definition 2.2(a)). As in example 4.4, $E=\{0,-\frac{1}{2}\}$ is an $\omega-$independent subset of dual Chebychev polynomial hypergroup of the first kind, and the center of this hypergroup is $\{+1,-1\}$. So in general an $\omega-$independent set does not necessarily intersect the center of the hypergroup. Theorem 2.9 implies that in some hypergroups there are infinite $\omega-$independent sets that are not independent. Then $\omega-$independence property properly contains independence property (see proposition 2.3).

One of the main results of the section 2 is that every $\omega-$independent subset of Γ with elements of infinite order is a Sidon set. (A subset $E\subseteq K$ is called a Sidon set if there is a constant B such that $\sum_{\gamma\in\Gamma}|\hat{f}(\gamma)|\le B\|f\|_\infty$ for every E-polynomial f on K [19].) In the sequel we define and study the Kronecker sets on hypergroups and show that under certain conditions, an $\omega-$independent set is a Kronecker set. These sets have been defined and named K_0-sets by Hewitt and Kakutani [8] and then Rudin called them Kronecker sets on groups [16]. In particular, most recently ε-Kronecker and I_0 sets in abelian groups have been studied in a series of papers [5-7]. We go further and introduce dissociate sets on duals of hypergroups and by using the Riesz products for hypergroups [21] we are able to prove some results on bounded dissociate sets. As a corollary we see that every bounded strongly asymmetric dissociate set is a Sidon set.

In the final section we bring some examples and find $\omega-$independent and dissociate sets on a compact countable hypergroup of Dunkle and Ramirez [3], the dual Chebychev polynomial hypergroup, and some other polynomial hypergroups from Lasser [13].

The main references for definitions and properties of hypergroups are [10] and [1]; see also [14].

Notations. Let K be a locally compact Hausdorff space. We denote by $M(K)$ the space of all finite Borel measures on K, by $M^+(K)$ the set of all positive measures in $M(K)$, and by δ_x the Dirac measure at the point x. The space K is a hypergroup if there exists a binary mapping $(x,y)\mapsto\delta_x*\delta_y$ from $K\times K$ into $M^+(K)$ satisfying the following conditions,

(1) The mapping $(\delta_x,\delta_y)\mapsto\delta_x*\delta_y$ extends to a bilinear associative operator $*$ from $M(K)\times M(K)$ into $M(K)$ such that

$$\int_K fd(\mu*\nu)=\int_K\int_K\int_K fd(\delta_x*\delta_y)d\mu(x)d\nu(y)$$

for all continuous functions f in $C_0(K)$.

(2) For each $x,y\in K$, the measure $\delta_x*\delta_y$ is a probability measure with compact support.

(3) The mapping $(\mu,\nu)\mapsto\mu*\nu$ is continuous from $M^+(K)\times M^+(K)$ into $M^+(K)$; the topology on $M^+(K)$ being the cone topology.

(4) There exists $e\in K$ such that $\delta_e*\delta_x=\delta_x=\delta_x*\delta_e$ for all $x\in K$.

(5) There exists a homeomorphism involution $x\mapsto x^-$ from K onto K such that, for all $x,y\in K$, we have $(\delta_x*\delta_y)^-=\delta_{y^-}*\delta_{x^-}$ where for $\mu\in M(K)$, μ^- is defined by

$$\int_K f(t)d\mu^-(t)=\int_K f(t^-)d\mu(t),$$

and also,

$$e\in supp(\delta_x*\delta_y)\text{ if and only if }y=x^-,$$

where $supp(\delta_x*\delta_y)$ is the support of the measure $\delta_x*\delta_y$.

(6) The mapping $(x,y)\mapsto supp(\delta_x*\delta_y)$ is continuous from $K\times K$ into the space $\mathbf{C}(K)$ of compact subsets of K, where $\mathbf{C}(K)$ is given the topology whose sub-basis is given by all

$$\mathbf{C}_{UV}=\{A\in\mathbf{C}(K):A\cap U\neq\emptyset\text{ and }A\subseteq V\}$$

where U, V are open subsets of K.

Note that $\delta_x*\delta_y$ is not necessarily a Dirac measure. The set $Z(K):=\{x\in K:$ for all $y\in K$, $supp(\delta_x*\delta_y)$ is a singleton$\}$ is called the center of hypergroup. Then $Z(K)$ with $(x,y)\mapsto x*y$ is a locally compact semigroup and a group if K is commutative ($x*y=z$ means $\delta_x*\delta_y=\delta_z$, where $x,y\in Z(K)$). A hypergroup K is commutative if $\delta_x*\delta_y=\delta_y*\delta_x$ for all x,y in K. Let us first recall some properties of locally compact commutative hypergroups. Such a hypergroup K carries a left Haar measure m such that $\delta_x*m=m$ for all $x\in K$ [18].

If $x, y \in K$, for a Borel function f on K, $f(x * y) = \int_K fd(\delta_x * \delta_y)$. A complex-valued continuous function ξ on K is said to be multiplicative if $\xi(x * y) = \xi(x)\xi(y)$ holds for all $x, y \in K$. A multiplicative function ξ on K is called a character if $\xi(x^-) = \overline{\xi(x)}$ for all x in K. The dual $\Gamma = \hat{K}$ of K is the space of all characters of K. Γ is not necessarily a hypergroup. A hypergroup K is called *strong* if its dual Γ is also a hypergroup with complex conjugation as involution, pointwise product as convolution, that is

$$\eta(x)\chi(x) = \int_{\hat{K}} \xi(x) d\, \delta_\eta * \delta_\chi(\xi)$$

for all $\eta, \chi \in \hat{K}$ and $x \in K$, and the constant function **1** as the identity element, and $\hat{\Gamma} \simeq K$, where $\hat{\Gamma}$ is the dual of Γ.

For $\mu \in M(K)$, the Fourier-Stieltjes transform $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(\xi) = \int_K \overline{\xi(t)} d\mu(t), (\xi \in \Gamma).$$

Similarly the Fourier transform $\hat{f}$ of a function $f \in L^1(K)$ is defined as

$$\hat{f}(\xi) = \int_K \overline{\xi(t)} f(t) dm(t),$$

where $\xi \in \Gamma$.

Throughout this paper K is a compact commutative strong hypergroup.

For a subset E of K, the closed subhypergroup generated by E i.e., the intersection of all closed subhypergroups of K contains E, is denoted by $\langle E \rangle$. If $x, y \in K$ and $A, B \subseteq K$ we denote $\{x\} * \{y\} = supp(\delta_x * \delta_y)$, $A * B = \bigcup_{x \in A, y \in B} \{x\} * \{y\}$, $\{x\}^0 = \{e\}$, $\{x\}^n = \{x\} * \{x\}^{n-1}$ and $\{x\}^{-n} = \{x^-\}^n$ for a positive integer n. The sets of natural numbers, integers, non-negative integers, non-positive integers, and complex numbers with absolute value one are denoted by N, Z, Z^+, Z^- and T, respectively. Then for every subset $o(x)$ of $n + n'$,

$$\langle E \rangle = \bigcup \{ \{\xi_1\}^{n_1} * \{\xi_1\}^{n_1'} * \ldots * \{\xi_k\}^{n_k} * \{\xi_k\}^{n_k'} : n_i \in Z^+,$$

$$n_i' \in Z^-,\ k \in N,\ \xi_i \in E \}.$$

Almost periodic functions on hypergroups have been studied in [11]. We denote the set of all almost periodic functions on $m' \in Z^-$ by $k = m + m'$ and by $e \in \{x\}^m * \{x\}^{m'}$ the space of all bounded linear operators on $AP(\Gamma)$ with strong operator topology. The function $n + n' > 0$ is defined by r, s, where $0 \le s < r$ ($n + n' = rk + s$). $e \in \{x\}^m * \{x\}^{m'}$ is not an algebra in general. In fact if $e \in \{x\}^n * \{x\}^{n'}$ is a hypergroup compactification of $0 \le s < r$ then $(-rm + n')$ $+(-rm' + n) = s$ is an algebra and $o(x)$ (for more details we refer to [11]).

2. s = 0 Independent and Kronecker Sets

First we extend the usual definition of order of an element of a group to elements of a hypergroup.

Definition 2.1. Let $o(x)$. If there exist $n \in Z^+$ and $n' \in Z^-$ such that $o(e) = 1$ and $o(x) = o(x^-)$, then we define the order of $o(a) = 1$ by

$$o(x) = \min\{|n + n'| : n \in Z^+, \ n' \in Z^-,$$

$$n + n' \ne 0, e \in \{x\}^n * \{x\}^{n'}\}$$

Otherwise we say K has infinite order and denote $\omega -$.

Remark. As in the group case it is not difficult to see that if $n \in Z^+$, $n' \in Z^-$, $n + n' \ne 0$ and $e \in \{x\}^n * \{x\}^{n'}$ then $o(x)$ divides $n + n'$. For, if $o(x) = k$ (k is nonzero), then by above definition there exist $m \in Z^+$ and $m' \in Z^-$ such that $k = m + m'$ and $e \in \{x\}^m * \{x\}^{m'}$. If $n + n' > 0$, then by division algorithm there exist non-negative integers r, s such that $0 \le s < r$ and $n + n' = rk + s$. Since $e \in \{x\}^m * \{x\}^{m'}$ and $e \in \{x\}^n * \{x\}^{n'}$ we have $e \in \{x\}^{-rm} * \{x\}^{-rm'} * \{x\}^n * \{x\}^{n'} = \{x\}^{-rm'+n} * \{x\}^{-rm+n'}$. But $0 \le s < r$ and $(-rm + n') + (-rm' + n) = s$, so by definition of $o(x)$, $s = 0$. That is $o(x)$ divides $n + n'$.

Also we can easily see that $o(e) = 1$ and $o(x) = o(x^-)$. But there are some differences too. For instance, in the finite hypergroups 9.1B, 9.1C and 9.1D of [10] we have $o(a) = 1$, while a is not the identity e.

If K is a locally compact commutative group the following definitions of $\omega -$ independent and Kronecker sets correspond to the definitions of independent and Kronecker sets in the group case given in [9].

Definition 2.2. (a) A subset E of K is called

$\omega-$independent set if for every choice of distinct elements $x_1,...,x_k$ of E, non-negative integers $n_1,...,n_k$ and non-positive integers $n'_1,...,n'_k$, the inclusion $e \in \{x_1\}^{n_1} * \{x_1\}^{n'_1} * ... * \{x_k\}^{n_k} * \{x_k\}^{n'_k}$ implies $e \in \{x_i\}^{n_i} * \{x_i\}^{n'_i}$ ($i = 1,...,k$).

(b) A subset E of K is called a Kronecker set if for every continuous function f on E with $|f| = 1$ and every $\varepsilon > 0$, there exists $\xi \in \Gamma$ such that $|f(x) - \xi(x)| < \varepsilon$ ($x \in E$).

Proposition 2.3. Every independent subset of Γ is also $\omega-$independent.

Proof. Let E be an independent subset of Γ. Then $1 \notin E$, and for any finite subset $F = \{\xi_1,...,\xi_k\}$ of E:

(i) for each $\xi \in E \setminus F$, $\langle F \rangle \bigcap \langle \xi \rangle = \{1\}$; and

(ii) for all $\psi_i \in \langle \xi_i \rangle$ ($i = 1,...,k$), $\{\psi_1\} * ... * \{\psi_k\}$ is a singleton in Γ.

Let $1 \in \{\xi_1\}^{n_1} * \{\xi_1\}^{n'_1} * ... * \{\xi_k\}^{n_k} * \{\xi_k\}^{n'_k}$ where $n_1,...,n_k \in \mathbb{Z}^+$ and $n'_1,...,n'_k \in \mathbb{Z}^-$. Then for some (unique) $\psi_i \in \{\xi_i\}^{n_i} * \{\xi_i\}^{n'_i}$ ($i = 1,...,k$) we have $1 \in \{\psi_1\} * ... * \{\psi_k\}$ and by (ii), $\{1\} = \{\psi_1\} * ... * \{\psi_k\}$. For any subsets A, B of Γ, $1 \in A^- * B$ if and only if $A \bigcap B \neq \varnothing$ [7, 4.1.A]. Hence $\psi^- \in \{\psi_2\} * ... * \{\psi_k\}$, and so by ($i$), $\psi_1 = 1$. Similarly $\psi_i = 1$ for $i = 2,..,k$ and the proof is completed. □

We have an interesting result on real hypergroups.

Theorem 2.4. For $K := \mathbb{R}$ or $[0,1]$ let $(K,*)$ be a hypergroup such that for every $x, y \in K$, $supp(\delta_x * \delta_y)$ contains at most two points. Then $(K,*)$ contains an infinite $\omega-$independent set.

Proof. In the case $K = \mathbb{R}$, $(K,*)$ is isomorphic to one of the following hypergroups:

(i) $cosh$-hypergroup $\mathbb{R}^+$ together with the convolution defined by

$$\delta_r * \delta_s = \frac{cosh(r-s)}{2coshr.coshs}\delta_{|r-s|} + \frac{cosh(r+s)}{2coshr.coshs}\delta_{r+s},$$

where $r, s \in \mathbb{R}^+$, and identity as involution; or

(ii) hermitian one-dimensional hypergroup $(\mathbb{R}^+,*)$ with convolution $\delta_r * \delta_s = \frac{1}{2}\delta_{|r-s|} + \frac{1}{2}\delta_{r+s}$, [1, Theorem 3.4.28].

In the case $K = [0,1]$, $(K,*)$ is isomorphic to

(iii) the (double-coset) hermitian one-dimensional hypergroup $([0,1],*)$ with convolution given by $\delta_r * \delta_s = \frac{1}{2}\delta_{|r-s|} + \frac{1}{2}\delta_{|1-|r-s||}$ where $r, s \in [0,1]$, [1, Theorem 3.4.21].

Hence it is enough to establish the existence of infinite $\omega-$independent sets in these three hypergroups.

In the both hypergroups (i) and (ii), $\{r\} * \{s\} = \{|r-s|, r+s\}$ ($r, s \in \mathbb{R}^+$). Then we can easily see that for every $r, s \in \mathbb{R}^+$, $\{r\}^m = \{0, 2r, 4r, ..., mr\}$ and $\{s\}^n = \{s, 3s, 5s, ..., ns\}$ and also $0 \in \{r\}^m * \{s\}^n$ if and only if $\frac{r}{s} \in \{\frac{1}{2}, \frac{3}{2}, ..., \frac{n}{2}, \frac{1}{4}, \frac{3}{4}, ..., \frac{n}{4}, ..., \frac{1}{m}, \frac{3}{m}, ..., \frac{n}{m}\}$, where n, m are odd and even integers, respectively. Similarly for m, n odd, $0 \in \{r\}^m * \{s\}^n$ if and only if $\frac{r}{s} \in \{1, 3, ..., n, \frac{1}{3}, \frac{3}{3}, ..., \frac{n}{3}, \frac{1}{5}, \frac{3}{5}, ..., \frac{n}{5}, ..., \frac{1}{m}, \frac{3}{m}, ..., \frac{n}{m}\}$. So if at least one of the integers $n_1,...,n_k$ is odd, we have $0 \in \{r_1\}^{n_1} * ... * \{r_k\}^{n_k}$ if and only if there exists $(a_1,...,a_k) \neq (0,...,0)$ in $\mathbb{Z}^k$ such that $a_1 r_1 + ... + a_k r_k = 0$. Therefore for any non-negative non-algebraic number $\zeta \in \mathbb{R}$ and every distinct elements $r_1,...,r_k$ in $S = \{\zeta, \zeta^2, \zeta^3, ...\}$, $0 \in \{r_1\}^{n_1} * ... * \{r_k\}^{n_k}$ if and only if $n_1,...,n_k$ are all even. So $0 \in \{r_1\}^{n_1} * ... * \{r_k\}^{n_k}$ implies $0 \in \{r_1\}^{n_1}, ..., 0 \in \{r_k\}^{n_k}$, and this shows that S is $\omega-$independent.

In the hypergroup (iii), every element of $\{r\}^n$ is of the form $a + br$, where $r \in [0,1]$, $n \in \mathbb{Z}^+$, $a, b \in \mathbb{Z}$. In general, every element of $\{r_1\}^{n_1} * ... * \{r_k\}^{n_k}$ is a linear combination of $r_1,...,r_k$ with integer coefficients. This implies that for any non-algebraic $\eta \in [0,1]$ and odd integer n we have $0 \notin \{\eta\}^n$. Hence for every distinct elements $r_1,...,r_k$ in $T = \{\eta, \eta^2, \eta^3, ...\}$ and integers $n_1,...,n_k$, $0 \in \{r_1\}^{n_1} * ... * \{r_k\}^{n_k}$ if and only if $n_1,...,n_k$ are all even and so $0 \in \{r_i\}^{n_i}$ ($i = 1,...,k$). Therefore T is $\omega-$independent. □

Remark. The sets S and T introduced in the proof of theorem 2.4 do not satisfy in the condition (ii) of independent sets and so are not independent.

Theorem 2.5. Every $\omega-$independent set $E \subseteq \Gamma$ with elements of infinite order is a Sidon set.

Proof. We will prove the condition (v) of the equivalence 2.2 in [19]. For this, let $F = \{\gamma_1,...,\gamma_k\}$ be a finite subset of E and consider

$f=\sum_{i=1}^{k}a_i\gamma_i\in Trig_E(K)$, where $a_1,\dots,a_n$ are complex numbers and $Trig_E(K)$ denotes all the trigonometric polynomials on E. Then by orthogonality of characters

$$|\hat{f}(\xi)|=\begin{cases}|a_i|K_{\gamma_i} & \text{if } \xi=\gamma_i\ (i=1,\dots,n),\\ 0 & \text{otherwise,}\end{cases}$$

where $K_\gamma=\int\gamma\bar{\gamma}dm$. We have

$$K_\gamma=\int\gamma(x)\bar{\gamma}(x)dm(x)$$
$$=\iint\xi(x)dm(x)d\delta_\gamma*\delta_{\bar{\gamma}}(\xi)=(\delta_\gamma*\delta_{\bar{\gamma}})(\{1\}),$$

and so

$$\|\hat{f}\|_1=\int|\hat{f}(\xi)|dm(\xi)=\sum_{i=1}^{n}\frac{|a_i|K_{\gamma_i}}{(\delta_{\gamma_i}*\delta_{\bar{\gamma}_i})(\{1\})}=\sum_{i=1}^{n}|a_i|.$$

Put $A_1=\sum_{Re(a_i)\ge0}Re(a_i)$, $A_2=\sum_{Re(a_i)<0}|Re(a_i)|$, $A_3=\sum_{Im(a_i)\ge0}Im(a_i)$ and $A_4=\sum_{Im(a_i)<0}|Im(a_i)|$. Then

$$\sum_{i=1}^{n}|a_i|\le\sum_{i=1}^{n}|Re(a_i)|+|Im(a_i)|\le 4.max\{A_1,\dots,A_4\}.$$

Let $A_1=max\{A_1,\dots,A_4\}$, (proofs in the other cases are similar). For $i=1,\dots,n$, if $Re(a_i)\ge0$ we take $x_i=e$. If $Re(a_i)<0$, since $\int\gamma_i dm=0$ and $\gamma_i(e)=1$, there exists some v in K such that $Re(\gamma_i(v))<0$. Choose $x_i\in\{v,v^-\}$ such that $Im(a_i)$ and $Im(\gamma_i(x_i))$ have different signs. So if $Re(a_i)\ge0$ then $Re(a_i.\gamma_i(x_i))=Re(a_i)$, and if $Re(a_i)<0$ then

$$Re(a_i.\gamma_i(x_i))=Re(a_i)Re(\gamma_i(x_i))$$
$$-Im(a_i)Im(\gamma_i(x_i))\ge0.$$

Hence

$$A_1\le Re(\sum_{i=1}^{n}a_i.\gamma_i(x_i)).$$

Claim: there exists an $x\in K$ such that $\gamma_i(x_i)=\gamma_i(x)$, for all $i=1,\dots,k$.

To prove this claim, first note that the function φ defined on $\langle F\rangle$ by $\varphi(\xi)=\prod_{i=1}^{k}(\gamma_i(x_i))^{n_i+n'_i}$, where $n_i\in\mathbb{Z}^+$, $n'_i\in\mathbb{Z}^-$ $(i=1,\dots,k)$ and $\xi\in\{\gamma_1\}^{n_1}*\{\gamma_1\}^{n'_1}*\dots*\{\gamma_k\}^{n_k}*\{\gamma_k\}^{n'_k}$, is well-defined, because if for other integers $m_i\in\mathbb{Z}^+$ and $m'_i\in\mathbb{Z}^-$ we also have $\xi\in\{\gamma_1\}^{m_1}*\{\gamma_1\}^{m'_1}*\dots*\{\gamma_k\}^{m_k}*\{\gamma_k\}^{m'_k}$ then

$$1\in\{\gamma_1\}^{n_1}*\{\gamma_1\}^{n'_1}*\dots*\{\gamma_k\}^{n_k}*\{\gamma_k\}^{n'_k}*\{\gamma_1\}^{-m_1}$$
$$*\{\gamma_1\}^{-m'_1}*\dots*\{\gamma_k\}^{-m_k}*\{\gamma_k\}^{-m'_k}$$
$$=\{\gamma_1\}^{n_1-m'_1}*\{\gamma_1\}^{n'_1-m_1}*\dots*\{\gamma_k\}^{n_k-m'_k}*\{\gamma_k\}^{n'_k-m_k},$$

and then, since E is $\omega-$independent, $1\in\{\gamma_i\}^{n_i-m'_i}*\{\gamma_i\}^{n'_i-m_i}$ for $i=1,\dots,k$. Now because every element of E is of infinite order, we have $m_i-n'_i+m'_i-n_i=0$ for $i=1,\dots,k$. In addition, for given ξ,η in $\langle F\rangle$ that $\xi\in\{\gamma_1\}^{n_1}*\{\gamma_1\}^{n'_1}\dots*\{\gamma_k\}^{n_k}*\{\gamma_k\}^{n'_k}$ and $\eta\in\{\gamma_1\}^{m_1}*\{\gamma_1\}^{m'_1}\dots*\{\gamma_k\}^{m_k}*\{\gamma_k\}^{m'_k}$, where $n_i,m_i\in\mathbb{Z}^+$ and $n'_i,m'_i\in\mathbb{Z}^-$, we have $\{\xi\}*\{\eta\}\subseteq\{\gamma_1\}^{n_1+m_1}*\{\gamma_1\}^{n'_1+m'_1}*\dots*\{\gamma_k\}^{n_k+m_k}*\{\gamma_k\}^{n'_k+m'_k}$ and then

$$\varphi(\xi*\eta)=\int\varphi(t)d(\delta_\xi*\delta_\eta)(t)$$
$$=\int\prod_{i=1}^{k}(\gamma_i(x_i))^{n_i+m_i+n'_i+m'_i}d(\delta_\xi*\delta_\eta)(t)$$
$$=\prod_{i=1}^{k}(\gamma_i(x_i))^{n_i+m_i+n'_i+m'_i}=\varphi(\xi)\varphi(\eta).$$

Hence φ is a character on $\langle F\rangle$. Since $\hat{K}$ is also a hypergroup and $\hat{\hat{K}}\simeq K$, K separates the elements of $\langle F\rangle$. Therefore there exists an $x\in K$ such that $\gamma_i(x)=\varphi(\gamma_i)=\gamma_i(x_i)$ and the claim is proved.

Thus $Re(\sum_{i=1}^{n}a_i.\gamma_i(x))=Re(f(x))$ and then

$$\|\hat{f}\|_1\le4\cdot Re(f(x))\le4\cdot|f(x)|\le4\cdot\|f\|_\infty. \qquad \square$$

Theorem 2.6. Suppose Γ^a is a hypergroup. Every finite $\omega-$independent subset E of Γ that generates Γ and has no element of finite order is a Kronecker set.

Proof. Let f be a function on E with $|f|=1$ and $E=\{\xi_1,\dots,\xi_k\}$. Since Γ is generated by E, for every $\xi\in\Gamma$ there exist integers $n_i\in\mathbb{Z}^+$, $n'_i\in\mathbb{Z}^-$ $(i=1,\dots,k)$ such that $\xi\in\{\xi_1\}^{n_1}*\{\xi_1\}^{n'_1}*\dots*\{\xi_k\}^{n_k}*\{\xi_k\}^{n'_k}$. We define $\varphi:\Gamma\to T$ by $\varphi(\xi)=\prod_{i=1}^{k}f(\xi_i)^{n_i+n'_i}$. Note that $\varphi(\xi_i)=f(\xi_i)$ for $i=1,\dots,k$. If also for other $m_i\in\mathbb{Z}^+$, $m'_i\in\mathbb{Z}^-$ $(i=1,\dots,k)$, $\xi\in\{\xi_1\}^{m_1}*\{\xi_1\}^{m'_1}*\dots*\{\xi_k\}^{m_k}$

$*\{\xi_k\}^{m'_k}$ then in the same way as the function φ defined in the proof of theorem 2.4, φ is well-defined. Also φ is multiplicative. Because if $\xi \in \{\xi_1\}^{n_1} * \{\xi_1\}^{n'_1} * ... * \{\xi_k\}^{n_k} * \{\xi_k\}^{n'_k}$ and $\eta \in \{\xi_1\}^{m_1} * \{\xi_1\}^{m'_1} * ... * \{\xi_k\}^{m_k} * \{\xi_k\}^{m'_k}$, where $n_i, m_i \in \mathbb{Z}^+$, $n'_i, m'_i \in \mathbb{Z}^-$, then

$$\{\xi\} * \{\eta\} \subseteq \{\xi_1\}^{n_1+m_1} * \{\xi_1\}^{n'_1+m'_1} * ... * \{\xi_k\}^{n_k+m_k} * \{\xi_k\}^{n'_k+m'_k},$$

and therefore

$$\varphi(\xi * \eta) = \int_{\{\xi\}*\{\eta\}} \varphi(\gamma) d(\delta_\xi * \delta_\eta)(\gamma)$$
$$= \int_{\{\xi\}*\{\eta\}} \prod_{i=1}^{k} f(\xi_i)^{n_i+m_i+n'_i+m'_i} d(\delta_\xi * \delta_\eta)(\gamma)$$
$$= \prod_{i=1}^{k} f(\xi_i)^{n_i+m_i+n'_i+m'_i} = \varphi(\xi)\varphi(\eta).$$

Since Γ^a is a hypergroup, $AP(\Gamma)$ is an algebra and $A(\Gamma^a) = C(\Gamma^a)$ [11]. So $\Gamma^a \cong \Delta(A(\Gamma^a)) \cong \Delta(C(\Gamma^a))$ (for a commutative Banach algebra A, $\Delta(A)$ denotes the maximal ideal space of A). Therefore for every $\varepsilon > 0$ there exists $x_0 \in K$ such that $|\varphi(\xi_i) - \xi_i(x_0)| < \varepsilon$ ($i = 1, ..., k$). □

3. Dissociate Sets

Dissociate sets are weak versions of independent sets that were introduced in [9] for characters group of a compact abelian group. We generalize this concept for duals of compact hypergroups and give some results on hypergroups. Throughout this section K is a compact and strong hypergroup (that is, Γ is also a hypergroup). A subset E of Γ is called bounded if $\{K_\gamma : \gamma \in E\}$ is bounded, where $K_\gamma^{-1} = \int \gamma\overline{\gamma} dm$. The subset E is called symmetric if $E = \{\overline{\gamma} : \gamma \in E\}$, is called asymmetric if $1 \notin E$ and $\gamma \in E$ with $\gamma \neq \overline{\gamma}$ imply $\overline{\gamma} \notin E$, and is called strong asymmetric if for all $\gamma \in E$ we have $\gamma \neq \overline{\gamma}$ and $\overline{\gamma} \notin E$. Obviously strongly asymmetric implies asymmetric.

Definition 3.1. A subset E of Γ is said to be dissociate if $1 \notin E$ and for every distinct elements $\gamma_1, ..., \gamma_k \in E$

(i) $\{\gamma_1\}^{\varepsilon_1} * ... * \{\gamma_k\}^{\varepsilon_k}$ is a singleton for each $\varepsilon_1, ..., \varepsilon_k \in \{+1, -1\}$;

(ii) for each $n_i \in \mathbb{Z}^+$ and $n'_i \in \mathbb{Z}^-$ with $n_i + n'_i \in \{0, \pm1, \pm2\}$ ($i = 1, ..., k$), if $1 \in \{\gamma_1\}^{n_1} * \{\gamma_1\}^{n'_1} * ... * \{\gamma_k\}^{n_k} * \{\gamma_k\}^{n'_k}$ then $1 \in \{\gamma_i\}^{n_i} * \{\gamma_i\}^{n'_i}$ ($i = 1, ..., k$);

(iii) E is asymmetric.

Remark. As in the example 4.4 a dissociate set dose not necessarily belong to the center of the hypergroup. Moreover if K is a compact abelian group then the definition of dissociation given above corresponds to the usual definition of dissociation in groups. For $s > 0$ and any subset N of Γ not containing **1** define

$$R_s(N, \gamma) = \sum_B (\prod_{\psi \in B} \delta_\psi)(\gamma)$$

where the sum is over all asymmetric subsets B of $N \bigcup \{\overline{\gamma} : \gamma \in N\}$ with $card(B) = s$. A subset E of Γ is called quasi-independent if for every finite subset A of E, $R_s(A, 1) = 0$ for $s = 1, 2, ...$ [21]. Suppose E is a dissociate subset of Γ and $N = E \setminus \{\gamma : \gamma = \overline{\gamma}\}$. For $s = 1, 2, ...$ we put

$$R_s^2(N, \gamma) = \sum_B (\prod_{\psi \in B} \delta_\psi)(\gamma)$$

where the sum is over all the subsets B of $N \bigcup \{\overline{\gamma} : \gamma \in N\}$ with $card(B) = s$ and could be repeated in $\prod_{\psi \in B} \delta_\psi$ up to 2. Obviously $R_s^2(N, 1) = 0$ for $s = 1, 2, ...$. Thus every strongly asymmetric dissociate set is quasi-independent. Also we can easily see that every independent set is a dissociate set. So our definition of dissociation has the same position as dissociation does in groups.

The following interesting theorem is an extension of [6, 37.14] to hypergroups, with completely different technique.

Theorem 3.2. Let E be a strongly asymmetric dissociate subset of Γ bounded by $\mathbf{D}$, and g be a complex-valued hermitian function on $Q = E \bigcup \{\overline{\gamma} : \gamma \in E\}$ such that for every $\gamma \in E$, $|g(\gamma)| \leq \frac{1}{2\mathbf{D}}$. Then there exists a measure $\mu \in M^+(K)$ such that $\hat{\mu}(1) = 1$ and $\hat{\mu}(\gamma) = g(\gamma)$ for all $\gamma \in E$.

Proof. The function $h : Q \to \mathbb{C}$ defined by $h(\gamma) = K_\gamma \cdot g(\gamma)$, is clearly hermitian and $|h(\gamma)| \leq K_\gamma \cdot \frac{1}{2\mathbf{D}} \leq \frac{1}{2}$ ($\gamma \in Q$). The equalities

$\overline{1+h(\overline{\gamma})\overline{\gamma}+h(\gamma)\gamma}=1+\overline{h(\overline{\gamma})}\gamma+\overline{h(\gamma)}\overline{\gamma}=1+h(\gamma)\gamma+h(\overline{\gamma})\overline{\gamma}$, and inequalities $|h(\overline{\gamma})\overline{\gamma}+h(\gamma)\gamma|\leq 2|h(\gamma)|\leq 1$ imply that $1+h(\overline{\gamma})\overline{\gamma}+h(\gamma)\gamma$ is real and non-negative. For a finite asymmetric subset A of Q we define $P_A=\prod_{\gamma\in A}(1+h(\overline{\gamma})\overline{\gamma}+h(\gamma)\gamma)$. Then P_A is real-valued and non-negative and by [21] $P_A=\sum_{\gamma\in\Gamma}C_A(\gamma)\gamma$, where $C_A(\gamma)=\sum_B(\prod_{\psi\in B}h(\psi))(\prod_{\psi\in B}\delta_\psi)(\gamma))$ (the last sum ranges over all subsets B of A).

For distinct elements $\gamma_1,...,\gamma_k\in E$ and numbers $\alpha_1,...,\alpha_k\in\{+1,-1\}$ let $\gamma=\{\gamma_1\}^{\alpha_1}*...*\{\gamma_k\}^{\alpha_k}$. If also for other distinct elements $\eta_1,...,\eta_t\in E$ and numbers $\beta_1,...,\beta_t\in\{+1,-1\}$, we have $\gamma=\{\eta_1\}^{\beta_1}*...*\{\eta_t\}^{\beta_t}$, then $1\in\{\gamma_1\}^{\alpha_1}*...*\{\gamma_k\}^{\alpha_k}*\{\eta_1\}^{-\beta_1}*...*\{\eta_t\}^{-\beta_t}$. If for an index j, $\eta_j\notin\{\gamma_1,...,\gamma_k\}$, $1\in\{\eta_j\}^{-\beta_j}$ and so $\eta_j=1$, a contradiction. Hence γ_i 's are unique. Now let $\eta_i=\gamma_i$ ($i=1,...,k$). Since E is dissociate and $\alpha_i-\beta_i\in\{0,\pm1,\pm2\}$ ($i=1,...,k$), $1\in\{\gamma_i\}^{\alpha_i}*\{\gamma_i\}^{-\beta_i}$. If $\alpha_i=1,\beta_i=-1$ (similarly if $\beta_i=1,\alpha_i=-1$) then $1\in\{\gamma_i\}*\{\gamma_i\}$ and so $\gamma_i=\overline{\gamma_i}$ a contradiction because E is strongly asymmetric. Hence α_i 's are also unique. For $\gamma=\{\gamma_1\}^{\alpha_1}*...*\{\gamma_k\}^{\alpha_k}$ we have $C_A(\gamma)=h(\gamma_1)^{\alpha_1}...h(\gamma_k)^{\alpha_k}$ and so $\hat{P}_A(\gamma)=C_A(\gamma)\hat{\gamma}(\gamma)$ $=h(\gamma_1)^{\alpha_1}...h(\gamma_k)^{\alpha_k}K_\gamma^{-1}$. By relation (2.2) in [21] $\hat{P}_A(1)=C_A(1)=1$. For other γ 's, $\hat{P}_A(\gamma)=0$. Since $\hat{P}_A$ is non-negative we have $\|\hat{P}_Am\|=\|\hat{P}_A\|_1$ $=\int_K\hat{P}_A\,dm=\hat{P}_A(1)=1$. If E is finite we put $\mu=P_Em$ and if E is infinite we consider the set $\{P_Am:\text{finite}\,A\subseteq E\}$. With directing this set with inclusion,

$$\lim_A\hat{P}_A(\gamma)=\begin{cases}h(\gamma_1)^{\alpha_1}...h(\gamma_k)^{\alpha_k}K_\gamma^{-1} & \text{if } \gamma=\{\gamma_1\}^{\alpha_1}*...*\{\gamma_k\}^{\alpha_k},\\ 1 & \text{if } \gamma=1,\\ 0 & \text{for others.}\end{cases}$$

By Alaoglu 's theorem the net $\{P_Am:\text{finite }A\subseteq E\}$ has a weak*-cluster point $\mu\in M^+(K)$ and so for a subnet $\{\mu_\alpha\}$ of $\{P_Am\}$, w*-$lim_\alpha\mu_\alpha=\mu$. Finally

$$\hat{\mu}(\gamma)=\int\overline{\gamma}d\mu=\lim_\alpha\int\overline{\gamma}d\mu_\alpha=\lim_\alpha\hat{\mu}_\alpha(\gamma)=$$

$$\begin{cases}h(\gamma_1)^{\alpha_1}...h(\gamma_k)^{\alpha_k}K_\gamma^{-1} & \text{if } \gamma=\{\gamma_1\}^{\alpha_1}*...*\{\gamma_k\}^{\alpha_k},\\ 1 & \text{if } \gamma=1,\\ 0 & \text{for others.}\end{cases}$$

In particular $\hat{\mu}(\gamma)=h(\gamma)K_\gamma^{-1}=g(\gamma)$ for all $\gamma\in E$.□

Corollary 3.3. Every bounded strongly asymmetric dissociate subset E of Γ is a Sidon set.

Proof. Let g be a complex-valued function on E bounded by **C**. Take a real number **D** greater than $2\mathbf{C}\cdot sup_{\gamma\in E}|g(\gamma)|$. Then $\mathbf{D}^{-1}\cdot Re(g)$ and $\mathbf{D}^{-1}\cdot Im(g)$ are functions on E satisfying in the hypothesis of theorem 3.2. So there are $\mu_1,\mu_2\in M^+(K)$ such that $\hat{\mu}_1=\mathbf{D}^{-1}\cdot Re(g)$ and $\hat{\mu}_2=\mathbf{D}^{-1}\cdot Im(g)$ on E. Thus $(D\cdot\mu_1+iD\cdot\mu_2)\hat{}=g$ on E. Now by theorem 2.2 of [19] E is a Sidon set. □

Another proof for corollary 3.3 follows from corollary 4.4 of [21].

Corollary 3.4. Let $E\subseteq\Gamma$ be a strongly asymmetric and dissociate set, bounded by **C**. Then $g:E\to\mathbb{C}$ is bounded if and only if g is the restriction to E of a bounded positive-definite function on Γ.

Proof. Suppose g is bounded and put $h=(2\mathbf{C}\cdot sup\{|g(\gamma)|:\gamma\in E\})^{-1}g$. Then $|h(\gamma)|\leq\frac{1}{2\mathbf{C}}$ and by theorem 3.2. there exists an $\upsilon\in M^+(K)$ such that $h(\gamma)=\hat{\upsilon}(\gamma)$ $(\gamma\in E)$. Therefore $g(\gamma)=\hat{\mu}(\gamma)$ $(\gamma\in E)$ where $\mu=2\mathbf{C}\cdot sup\{|g(\gamma)|:\gamma\in E\}\upsilon$, and since the Fourier transform of the measure μ is bounded and positive definite on Γ, the corollary is established. (The other part of the corollary is plain.) □

4. Examples

Example 4.1. Fix a prime p and let Δ_p denote the ring of p-adic integers. Every $x\in\Delta_p$ has a unique expansion $x=x_0+x_1p+x_2p^2+...$ where $x_i\in\{0,1,...,p-1\}$ ($i=0,1,2,...$), we refer to §10 of [9] for details. For $G=\Delta_p$, the group of p-adic integers, and $H=\{x\in\Delta_p:x_0\neq0\}$, the group of units in Δ_p, the countable compact hypergroup G^H is homeomorphic to $\mathbf{Z}_\infty^+$, the one-point compactification of $\mathbb{Z}^+$. Then $\mathbf{Z}_\infty^+$ is a hermitian hypergroup with ∞ as identity and with the following convolution,

$\delta_m * \delta_n = \delta_{min\{m,n\}}$ for $m \neq n$ and,

$$\delta_n * \delta_n = \frac{p-2}{p-1}\delta_n + \sum_{k=n+1}^{\infty} \frac{1}{p^{k-n}}\delta_k .$$

Then for every $m \in \mathbb{Z}^+$ and every integer $n \geq 2$,

$\{m\}^n = \{m, m+1, m+2, ..., \infty\}$.

If $m_1, m_2 \in \mathbb{Z}^+$ and $m_1 < m_2$ then for all integers $n \geq 2$, $\infty \in \{m_1\}^n * \{m_2\} = \{m_1, m_1+1, ..., \infty\}$. This implies that the only $\omega-$independent subsets of $\mathbf{Z}_\infty^+$ are singletons. In addition $\infty \in \{m\}^{-1} * \{m\}^2 = \{m\}^3$ implies that for every $m \in \mathbb{Z}^+$, $o(m) = 1$.

We can identify the dual hypergroup $\hat{\mathbf{Z}}_\infty^+$ with $\{\chi_n : n = 0,1,2,...\}$, where χ_n is a function on $\mathbf{Z}_\infty^+$ defined by

$$\chi_n(m) = \begin{cases} 1 & \text{if } m \geq n \text{ or } m = \infty, \\ \dfrac{-1}{p-1} & \text{if } m = n-1, \\ 0 & \text{if } m \leq n-2. \end{cases}$$

We have $\chi_n \chi_m = \chi_{max\{m,n\}}$, and

$$\chi_n^2 = \frac{1}{p^{n-1}(p-1)}\chi_0 + \sum_{k=1}^{n-1} p^{k-n}\chi_k + \frac{p-2}{p-1}\chi_n .$$

$\hat{\mathbf{Z}}_\infty^+$ is hermitian and so every subset of $\hat{\mathbf{Z}}_\infty^+$ is asymmetric and satisfies the condition (i) of definition 3.1, because for distinct elements $\chi_{n_1}, ..., \chi_{n_k} \in \hat{\mathbf{Z}}_\infty^+$ we have

$\{\chi_{n_1}\} * ... * \{\chi_{n_k}\} = \{\chi_{max\{n_1,...,n_k\}}\}$.

If $0 < n_1 < n_2$ then $\chi_0 \in \{\chi_{n_1}\} * \{\chi_{n_2}\}^2 = \{\chi_{n_1}\} * \{\chi_0, \chi_1, ..., \chi_{n_1}, ..., \chi_{n_2}\} = \{\chi_0, \chi_1, ..., \chi_{n_2}\}$ but $\chi_0 \notin \{\chi_{n_1}\}$. Hence the only dissociate subsets of $\hat{\mathbf{Z}}_\infty^+$ are singletons.

The following example comes from [13].

Example 4.2. Let $(a_n)_{n\in\mathbf{N}_0}, (c_n)_{n\in\mathbf{N}}$ be sequences of non-zero real numbers and $(b_n)_{n\in\mathbf{N}_0}$ be a sequence of real numbers with the following properties,

$a_0 + b_0 = 1$

$a_n + b_n + c_n = 1,\ n \geq 1.$

Let $(R_n)_{n\in\mathbf{N}_0}$ be a polynomial sequence defined by

$$R_0(x) = 1,\ R_1(x) = \frac{1}{a_0}(x - b_0),$$

$$R_1(x)R_n(x) = a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x),\ n \geq 1.$$

Then we have

$$R_n(x)R_n(x) = \sum_{k=|n-m|}^{n+m} g(n,m;k)R_k(x),$$

where $g(n,m;k) \in \mathbb{R}$ for all $|n-m| \leq k \leq n+m$. If the coefficients $g(n,m;k)$ satisfy

$g(n,m;k) \geq 0$ for all $n,m \in \mathbb{R}$

and $|n-m| \leq k \leq n+m$ $(\wp)$,

then $(\mathbb{N}_0, *)$ is a discrete commutative hermitian hypergroup (called a polynomial hypergroup) with convolution

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} g(n,m;k)\delta_k .$$

For details one can consult to [12].

Now fix $a,b > 2$ and put $a_0 = 1$, $a_1 = \frac{|a-1|}{a}$, $c_1 = \frac{1}{a}, a_2 = \frac{b-1}{b}, c_2 = \frac{1}{b}, a_n = c_n = \frac{1}{2}$ $(n \geq 3)$, $b_n = 0$ $(n \in \mathbb{N}_0)$. By recursion formula there exists an orthogonal polynomial $(T_n(x;a,b))_{n\in\mathbf{N}_0}$ satisfying property $(\wp)$. Using formula (1) of [12] $g(n,m;k)$'s can be computed and so that

$\{0\} * \{n\} = \{n\}$ $(n \geq 0)$,

$\{1\} * \{n\} = \{n-1, n+1\}$ $(n \geq 1)$,

$\{2\} * \{n\} = \{n-2, n, n+2\}$ $(n \geq 2)$,

$\{3\} * \{n\} = \{n-3, n-1, n+1, n+3\}$ $(n \geq 3)$,

$\{4\} * \{n\} = \{n-4, n-2, n, n+2, n+4\}$ $(n \geq 4)$,

$\{m\} * \{n\} = \{n-m, n-m+2, n-m+4, n+m-4, n+m-2, n+m\}$ $(n \geq m \geq 5)$.

Therefore

$\{1\}^n = \{0, 2, 4, ..., n\}$ (n even),

$\{1\}^n = \{1, 3, 5, ..., n\}$ (n odd),

$\{2\}^n = \{0,2,4,...,2n\}$ ($n \geq 2$),

$\{3\}^n = \{0,2,4,...,3n\}$ (n even),

$\{3\}^n = \{1,3,5,...,3n\}$ (n odd),

$\{4\}^n = \{0,2,4,...,4n\}$ ($n \geq 2$),

$\{5\}^n = \{0,2,4,...,5n\}$ (n even),

$\{5\}^n = \{1,3,5,...,5n\}$ (n odd),

$\{6\}^n = \{0,2,4,...,6n\}$ ($n \geq 4$),

$\{6\}^3 = \{2,4,6,8,10,12,14,16,18\}$,

$\{6\}^2 = \{0,2,4,8,10,12\}$,

$\{7\}^n = \{0,2,4,...,7n\}$ ($n \geq 4$ even),

$\{7\}^n = \{3,5,...,7n\}$ ($n \geq 3$ odd),

$\{7\}^2 = \{0,2,4,10,12,14\}$,

$\vdots$

Then for every odd $n \in \mathbb{N}$, the hypergroup $(\mathbb{N}_0,*)$ is generated by n, because $n \in \{n\}^3$; $n-2, n+2 \in \{n\}^5$; $n-4, n+4 \in \{n\}^7$ $1, 2n-1 \in \{n\}^{n+2}$; $2n+1 \in \{n\}^{n+4}$ For n even we have $2 \in \{n\}^2, 4 \in \{n\}^4, 6 \in \{n\}^6, \ldots$ and this implies that for every even numbers k, n there exists an m such that $k \in \{n\}^m$. Then by the following remark the only $\omega-$independent subsets of $(\mathbb{N}_0,*)$ are singletons.

Remark. Let $(K,*)$ be a discrete commutative hermitian hypergroup and let $A \subseteq K$. For every $x \neq e$ in $\langle A \rangle \setminus A$, the set $A \cup \{x\}$ is not $\omega-$independent. Because $x \in \langle A \rangle \setminus A$ implies that there are $a_1,...,a_k \in A$ and $n_1,...,n_k \in \mathbb{Z}^+$ such that $x \neq a_i$ ($i = 1,...,k$) and $x \in \{a_1\}^{n_1} * ... * \{a_k\}^{n_k}$. So $e \in \{x\} * \{a_1\}^{n_1} * ... * \{a_k\}^{n_k}$ and by $x \neq e$, $A \cup \{x\}$ is not $\omega-$independent.

Example 4.3. In some hypergroups $(\mathbb{N}_0,*)$ such as Chebyshev polynomial hypergroup of the first kind and also *cosh*-hypergroup on $\mathbb{N}_0$, we have $\{m\}*\{n\} = \{|n-m|, n+m\}$ $(m,n \in \mathbb{N}_0)$ [1]. So for every $n \in \mathbb{N}_0$,

$\{n\}^k = \{n, 3n, 5n, ..., kn\}$ (k odd) and $\{n\}^k$
$= \{0, 2n, 4n, ..., kn\}$ (k even).

Then $0 \in \{n\}^k * \{n\}^{k'} = \{n\}^{k-k'}$ ($k \in \mathbb{Z}^+$, $k' \in \mathbb{Z}^-$, $n \neq 0$) implies that $k - k'$ and so $k + k'$ is even. So the order of every element of $\mathbb{N}_0$ except 0 is 2. Let A be an $\omega-$independent subset of $\mathbb{N}_0$ and n_1, n_2 be distinct elements of A. Then there are $p, q \in \mathbb{N}$ such that $\gcd(p,q) = 1$ and $\frac{n_1}{n_2} = \frac{p}{q}$ and without lose of generality we may suppose that p is odd. The proof of theorem 2.4 implies that $0 \in \{n_1\}^q * \{n_2\}^p$, while $0 \notin \{n_2\}^p$, a contradiction. Hence the only $\omega-$independent subsets of $\mathbb{N}_0$ are singletons.

In the following example we find $\omega-$independent and dissociate sets that are not singleton.

Example 4.4. Let $G = \mathbf{T}$, the unit circle group, and $H = \{id, \tau\}$, where $id(x) = x$ and $\tau(x) = \bar{x}$ ($x \in \mathbf{T}$). Then $G^H \equiv [0,\pi]$ and if we replace $[0,\pi]$ with $[-1,1]$ by transformation $cos\,\theta = x$ then $[-1,1]$ is a hypergroup such that its structure is given by

$$\delta_x * \delta_y = \frac{1}{2}\delta_{xy - \sqrt{(1-x^2)(1-y^2)}} + \frac{1}{2}\delta_{xy + \sqrt{(1-x^2)(1-y^2)}}.$$

In fact it is dual Chebyshev polynomial hypergroup of the first kind [1]. The hypergroup $[-1,1]$ is hermitian and so the orders of it's elements are at most two. For every $x \in [-1,1]$, $\{x\}^2 = \{1, 2x^2 - 1\}$, $\{x\}^3 = \{x, 4x^3 - 3x\}$, $\{0\} * \{x\} = \{\pm\sqrt{1-x^2}\}$ and $\{-1\} * \{x\} = \{-x\}$. Also in particular $\{0\}^n = \{1,-1\}$ for $n \neq 0$ even, $\{0\}^n = \{0\}$ for n odd, $\{\frac{-1}{2}\}^n = \{1, \frac{-1}{2}\}$ for all $n \geq 2$, $\{-1\}^n = \{1\}$ for n even and $\{-1\}^n = \{-1\}$ for n odd. Then $o(0) = o(-1) = 2$ and $o(\frac{-1}{2}) = 1$. If $1 \in \{0\}^n * \{\frac{-1}{2}\}^m$ then $m, n > 1$ and n is even. So $1 \in \{0\}^n$ and $1 \in \{\frac{-1}{2}\}^m$. Therefore $E = \{0, \frac{-1}{2}\}$ is $\omega-$independent. For $x, y \in [-1,1]$, $\{x\} * \{y\}$ is singleton if $x = \pm 1$ or $y = \pm 1$. Then the subsets of the form $E = \{1, x\}$ or $E = \{-1, x\}$ satisfy the condition (i) of definition 3.1. We have $\{-1\}^n * \{x\}^m = \{x\}^m$ for n even and $\{-1\}^n * \{x\}^m = \{-t : t \in \{x\}^m\}$ for n odd. So if we take $x \in [-1,1]$ such that $-1 \notin \{x\}^m$ for all $m \in \mathbb{Z}^+$, then $E = \{-1, x\}$ $(x \neq 1)$ is a dissociate set because for n

odd $1 \notin \{-1\}^n * \{x\}^m$, and for n even $1 \in \{-1\}^n * \{x\}^m$ implies $1 \in \{-1\}^n$ and $1 \in \{x\}^m$. For instance $E = \{-1, \frac{-1}{2}\}$ is dissociate.

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