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Laws of Large Numbers for Random Linear Programs under Dependence Model

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Abstract

The computational solution of large scale linear programming problems contains various difficulties. One of the difficulties is to ensure numerical stability. There is another difficulty of a different nature, namely the original data, contains errors as well. In this paper, we show that the effect of the random errors in the original data has a diminishing tendency for the optimal value as the number of constraints and the number of variables increase. The laws of large numbers in probability theory are mathematical formulations for indicating the slowing-down tendency of the effect of random errors in the data. This paper was inspired by the paper of Prekopa [3]. Prekopa [3] proved both weak and strong laws of large numbers for the random linear programs in independence setting. We obtain laws of large numbers under negatively associated dependence for random linear programs and we extend Prekopa's results [3] to the case of negatively associated random variables.

Keywords: Negatively associated random variables; Random linear programs; Laws of large numbers

1. Introduction

It is well known that the computational solution of large scale linear programming problems contains various difficulties. One of the most difficult things is to ensure numerical stability, *i.e.*, to overcome the effect of round-off errors in the course of computation. Though this is a problematic feature of large systems, we have another difficulty of a different nature, namely that the original data we are using also contains errors. The question arises whether the effect of these latter type errors increase or decrease with the size of a random linear programming problem. We shall show that the effect of the random errors in the original data has a diminishing tendency for the optimal value as the number of constraints and the number of variables increase.

The laws of large numbers in probability theory are the mathematical formulations of the slowing-down tendency of the effect of random errors in the data when taking the arithmetic, or more generally, the weighted mean. Though many theorems (laws) are proved, we cannot expect to have a complete theory which would cover all practical situations. Similarly, we cannot expect that we can build up a "law of large numbers" theory for random linear programs containing direct answers to all special problems. At present the existence of the law of large numbers phenomenon is what we emphasize for dependence random variables in random linear programs and theorems are selected so that this will show up.

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This paper was inspired by the paper of Prekopa [3]. Prekopa [3] proved both weak laws and strong laws of large numbers for the random linear programs in independence setting. However, many variables are dependent in actual problems. For example, *negatively associated* random variables, its definition is as follows:

Definition. A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1,2,\ldots,n\}$,

$$\mathrm{Cov}\{f_1(X_i, i \in A), f_2(X_j, j \in B)\} \le 0,$$

whenever f_1 and f_2 are coordinatewise increasing and such that covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

This dependence structure was first introduced by Alam and Saxena [1] and carefully studied by Joag-Dov and Proschan [2].

Consider the following random linear programming problem:

$$\begin{aligned}
&\max\{c_1x_1 + c_2x_2 + \cdots + c_nx_n\}\\
&s.t.\\
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1\\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2\\
&\vdots\\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m\\
&x_1, x_2, \ldots, x_n \ge 0,
\end{aligned} \tag{1}$$

where the a_{ik}, $i = 1,2,\ldots,m$, $k = 1,2,\ldots,n$ are independent random variables, and $c_k > 0$, $k = 1,2,\ldots,n$ and b_i, $i = 1,2,\ldots,m$ are constants.

We may assume, without loss of generality, that $c_k = 1$, $k = 1,2,\ldots,n$, $b_i = 1$, $i = 1,2,\ldots,m$ because we can divide the i th constant by b_i, $i = 1,2,\ldots,m$ and introduce new variables, replacing c_kx_k by $x_k = 1$, $k = 1,2,\ldots,n$. If the optimum value of problem (1) is finite with probability 1, then it is also positive with probability 1.

Assuming the existence of the expectations $a_{ik}^{(0)} = E[a_{ik}]$, together with the random linear programming problem (1) we consider the deterministic problem:

$$\begin{aligned}
&\max\{c_1x_1 + c_2x_2 + \cdots + c_nx_n\}\\
&s.t.\\
&a_{11}^{(0)}x_1 + a_{12}^{(0)}x_2 + \cdots + a_{1n}^{(0)}x_n \le b_1\\
&a_{21}^{(0)}x_1 + a_{22}^{(0)}x_2 + \cdots + a_{2n}^{(0)}x_n \le b_2\\
&\vdots\\
&a_{m1}^{(0)}x_1 + a_{m2}^{(0)}x_2 + \cdots + a_{mn}^{(0)}x_n \le b_m\\
&x_1, x_2, \ldots, x_n \ge 0,
\end{aligned} \tag{2}$$

If the optimum value of problem (2) is finite, then it is also positive.

By independence assumption, Prekopa [3] proved, if $m, n \to \infty$ subject to some conditions, then the between the random optimum value μ of problem (1) and the optimum value $\mu^{(0)}$, that corresponds to the expectations, goes to 0 in probability or almost surely, depending on our conditions. In this note, we extend his results in the case of NA random variables.

The main theorems are proved in Section 2. In Section 1.1, we shall state some preliminaries which are important in the proof of the main results.

1.1. Preliminaries

It should be pointed out that, throughout this section, the letter C is used indiscriminately as a generic constant.

Lemma 1. (Shao [4]) Let $p > 2$, $\{\xi_i, 1 \le i \le n\}$ be a sequence of negatively associated mean zero random variables with $E|\xi_i|^p < \infty$ for every $1 \le i \le n$ and let $S_i = \sum_{j=1}^{i} \xi_j$. Then, we have

$$E(\max_{1 \le i \le n} |S_i|^p) \le C\{(\sum_{i=1}^{n} E\xi_i^2)^{p/2} + \sum_{i=1}^{n} E|\xi_i|^p\}. \qquad \blacksquare$$

The following corollary is a result of Lemma 1.

Corollary 1. Let $p > 2$, $\{\xi_i, 1 \le i \le n\}$ be a sequence of negatively associated mean zero random variables with $E|\xi_i|^p < \infty$ for every $1 \le i \le n$ and let $S_i = \sum_{j=1}^{i} \xi_j$. Then, we have

$$E|S_n|^p \le Cn^{p/2}. \qquad \blacksquare$$

Lemma 2. Let $\{\xi_{ik}, 1 \le i \le m, k = 1,2,\ldots,n\}$ be an array of mean zero random variables with $E|\xi_{ik}|^p \le K < \infty$ for $p > 2$ and satisfying the following conditions,

(i) $\{\xi_{ik}, k = 1,2,\ldots,n\}$ is a sequence of negatively associated for each $i = 1,2,\ldots,m$.

(ii) $\{\xi_{ik}, i = 1,2,\ldots,m\}$ is a sequence of negatively associated for each $k = 1,2,\ldots,n$.

(iii) Let $m, n \to \infty$ in such a way that the following condition is satisfied

$$0 < \alpha \le \frac{m}{n} \le \beta < \infty, \tag{3}$$

where α and β are constants. Under these conditions we have

$$\max_{1\le i\le m}\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}\to 0,\quad \textit{in probability}, \tag{4}$$

$$\min_{1\le k\le n}\frac{\xi_{1k}+\xi_{2k}+\cdots+\xi_{mk}}{m}\to 0,\quad \textit{in probability}. \tag{5}$$

Proof of Lemma 2. It is enough to prove that for any $\varepsilon>0$,

$$P(\max_{1\le i\le m}\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}\ge\varepsilon)\to 0, \tag{6}$$

whenever $m,n\to\infty$ while (3) holds. In fact if we apply (6) to the array ξ_{ik}, $1\le i\le m$, $k=1,2,\ldots,n$ so that ξ_{ik} is replaced by $-\xi_{ik}$, then we obtain

$$P(\min_{1\le i\le m}\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}\le -\varepsilon)\to 0. \tag{7}$$

(6) and (7) together imply (4). Applying (6) and (7) for the transpose of the array ξ_{ik}, $1\le i\le m$, $k=1,2,\ldots,n$, we obtain (5).

Let us now prove (6). By Corollary 1 we have,

$$\begin{aligned}
&P(\max_{1\le i\le m}\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}\ge\varepsilon)\\
&\le P(\max_{1\le i\le m}|\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}|\ge\varepsilon)\\
&\le \sum_{i=1}^{m}P(|\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}|\ge\varepsilon)\\
&= \sum_{i=1}^{m}P(|\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}|^p\ge\varepsilon^p)\\
&\le \sum_{i=1}^{m}\frac{1}{\varepsilon^p n^p}E\,|\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}|^p\\
&\le m\,\frac{1}{\varepsilon^p n^p}Cn^{p/2}\\
&\le n\beta\frac{1}{\varepsilon^p n^p}Cn^{p/2}\\
&= Cn^{1-p/2}.
\end{aligned}$$

This proves the lemma. ∎

Lemma 3. Under the assumptions of Lemma 2, for $p>6$, we have,

$$\max_{1\le i\le m}\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}\to 0\qquad a.s., \tag{8}$$

$$\min_{1\le k\le n}\frac{\xi_{1k}+\xi_{2k}+\cdots+\xi_{mk}}{m}\to 0\qquad a.s. \tag{9}$$

Proof of Lemma 3. We obtain similarly to the proof of Lemma 2, the following inequality

$$\begin{aligned}
&P(\max_{1\le i\le m}\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}\ge\varepsilon)\\
&\le\sum_{i=1}^{m}\frac{1}{\varepsilon^p n^p}E\,|\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}|^p,
\end{aligned}$$

where $\varepsilon>0$ is a fixed number. Applying Corollary 1, we conclude that

$$P(\max_{1\le i\le m}\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}\ge\varepsilon)\le Cn^{1-p/2}. \tag{10}$$

If we consider the sequence m,n as $m,n\to\infty$ if n is fixed, then the number of elements m,n in the sequence is at most $n\beta$ in view of (3). If we take into account this fact, (10) implies that

$$\sum_{m,n}P(\max_{1\le i\le m}\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}\ge\varepsilon)\le C\sum_{n=1}^{\infty}n^{2-p/2}<\infty.$$

By the Borel-Cantelli lemma it follows from this that except for at most a finite number of m,n we have

$$\max_{1\le i\le m}\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}<\varepsilon. \tag{11}$$

Applying this for the random variables $-\xi_{ik}$ instead of the random variables ξ_{ik}, we obtain that except for at most a finite number of m,n the following relation hold:

$$\min_{1\le i\le m}\frac{\xi_{i1}+\xi_{i2}+\cdots+\xi_{in}}{n}>-\varepsilon. \tag{12}$$

These imply (8). Interchanging m,n in (11) and (12) we see that (9) is also proved. This completes the proof of the lemma. ∎

2. Results

Now we turn to the proof of the laws of large numbers for random linear programs for NA random variables. In the sequel we will use the notation $\xi_{ik}=a_{ik}-a_{ik}^{(0)}$, $i=1,2,\ldots,m$, $k=1,2,\ldots,n$.

Theorem 1. Suppose that in connection with problem (1) the following conditions are satisfied:

(i) There exists positive integers m_0,n_0 such that for every $m\ge m_0,n\ge n_0$ the random linear programming problem (1) has a finite optimum value μ with probability 1; also, problem (2) has a finite optimum value $\mu^{(0)}$ and $\mu^{(0)}\le\delta$ where δ does not depend on m or n.

(ii) The random variables ξ_{ik}, $1 \le i \le m$, $k = 1,2,\ldots,n$ satisfy the conditions of Lemma 2.

(iii) For every $m \ge m_0$, $n \ge n_0$ problem (2) and its dual have an optimal solution pair $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})^T$, $y^{(0)} = (y_1^{(0)}, y_2^{(0)}, \ldots, y_m^{(0)})^T$ such that

$$\frac{nx_k^{(0)}}{x_1^{(0)} + x_2^{(0)} + \cdots + x_n^{(0)}} = \frac{nx_k^{(0)}}{\mu^{(0)}} \le L_1, \; i = 1,2,\ldots,m, \quad (13)$$

$$\frac{my_i^{(0)}}{y_1^{(0)} + y_2^{(0)} + \cdots + y_m^{(0)}} = \frac{my_i^{(0)}}{\mu^{(0)}} \le L_2, \; k = 1,2,\ldots,n, \quad (14)$$

where L_1, L_2 are constants. Then

$$\mu - \mu^{(0)} \to 0, \quad in\ probability \quad (15)$$

when $m, n \to \infty$.

Proof of Theorem 1. First we prove

$$\frac{1}{\mu} - \frac{1}{\mu^{(0)}} \to 0, \quad in\ probability. \quad (16)$$

Since the random linear programming problem (1) has a finite optimum value μ with probability 1, similar to the proof of equation (3.21) in Prekopa [3], we can get the following inequality easily;

$$\min_{1 \le k \le n} \sum_{i=1}^{m} \xi_{ik} \gamma_i^{(0)} \le \frac{1}{\mu} - \frac{1}{\mu^{(0)}} \le \max_{1 \le i \le m} \sum_{k=1}^{n} \xi_{ik} \lambda_k^{(0)}, \quad (17)$$

where

$$\lambda_k^{(0)} = \frac{x_k^{(0)}}{x_1^{(0)} + x_2^{(0)} + \cdots + x_n^{(0)}} = \frac{x_k^{(0)}}{\mu^{(0)}}, \; k = 1,2,\ldots,n,$$

and

$$\gamma_i^{(0)} = \frac{y_i^{(0)}}{y_1^{(0)} + y_2^{(0)} + \cdots + y_m^{(0)}} = \frac{y_i^{(0)}}{\mu^{(0)}}, \; i = 1,2,\ldots,m.$$

Relation (16) will be proved if we prove that both the left-hand side and the right-hand side of (17) tend to zero in probability. Let us consider the right-hand side of (17). Define the random variables

$$\eta_{ik} = \xi_{ik} n \lambda_k^{(0)}, \; i = 1,\ldots,m; \; k = 1,\ldots,n. \quad (18)$$

These random variables satisfy the conditions of Lemma 2 because the ξ_{ik} do and because (13) holds. Thus by Lemma 2 we have

$$\max_{1 \le i \le m} \sum_{k=1}^{n} \xi_{ik} \lambda_k^{(0)} = \max_{1 \le i \le m} \sum_{k=1}^{n} \eta_{ik} \to 0, \quad in\ probability.$$

The left hand side of (17) can be treated in an entirely similar way. Hence (16) follows. To prove (15) we mention first that by (i), there exists a constant M, such that for m, n large

$$P(\mu \le M) = 1. \quad (19)$$

We remark that $\mu \ge 0$. Thus (16), (19) and the boundedness of the sequence $\mu^{(0)}$ imply the limit relation (15). This completes the proof of Theorem 1. ■

Theorem 2. Under the assumptions of Theorem 1, for $p > 6$, we have,

$$\mu - \mu^{(0)} \to 0 \qquad a.s.,$$

when $m, n \to \infty$.

Proof of Theorem 2. The inequality (17) is satisfied with probability 1. On the other hand (i) ensure that

$$\max_{1 \le i \le m} \sum_{k=1}^{n} \xi_{ik} \lambda_k^{(0)} = \max_{1 \le i \le m} \frac{1}{n} \sum_{k=1}^{n} \eta_{ik} \to 0 \qquad a.s.,$$

where η_{ik} is obtained by (18). Thus the random variable on the right-hand side of (17) tends to 0, almost surely. The same reasoning can be applied to the left-hand side of (17). Hence

$$\frac{1}{\mu} - \frac{1}{\mu^{(0)}} \to 0 \quad a.s. \quad (21)$$

To prove (20), we remark that the boundedness of $\mu^{(0)}$ and the conditions of Theorem 1 are enough to derive (20). Thus Theorem 2 is proved.

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