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On Regularity of Acts

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Abstract

In this article we give a characterization of monoids for which torsion freeness, ((principal) weak, strong) flatness, equalizer flatness or Condition (E) of finitely generated and (mono) cyclic acts and Condition (P) of finitely generated and cyclic acts implies regularity. A characterization of monoids for which all (finitely generated, (mono) cyclic acts are regular will be given too. We also show that monoids for which all regular right acts are *WPF*, *WKF*, *PWKF*, *TKF* or satisfy Conditions (P), (WP) and (PWP) are the same as those for which all regular right acts are projective or strongly flat. Monoids S with $E(S)\subseteq C(S)$ or those for which every element of $E(S)\setminus\{1\}$ is right zero will be characterized, when all (finitely generated, cyclic) right S-acts satisfying Conditions (PWP), (WP), (P_E) or (P) are regular. Simple monoids for which all (finitely generated, cyclic) right acts with property U (U is a property of acts over monoids implied by Condition (P)) are regular will be characterized too.

Keywords: Regular act; Flatness; Right zero

Introduction

In [4] and [6] monoids for which all torsion free, ((principally) weakly, strongly) flat acts are regular were classified. But so far there is no classification of monoids when these flatness properties of (finitely generated, (mono) cyclic) acts are regular. In section 1 of this paper, we not only answer these questions, but also give a characterization of monoids for which equalizer flatness, Condition (P) and Condition (E) of finitely generated (cyclic) acts implies regularity.

We show that monoids for which all finitely generated (cyclic, monocyclic) right acts are regular, are the same as those for which all right acts are regular which characterized by Kilp and Knauer in [4].

Monoids for which all regular right S-acts are projective or strongly flat were considered by Kilp and Knauer in [4]. In this article we show these monoids are the same as those for which all regular right acts are *WPF*, *WKF*, *PWKF*, *TKF* or satisfy Conditions (P), (WP) and (PWP).

Finally, we consider monoids S with $E(S)\subseteq C(S)$ ($E(S)$ is the set of idempotents of S and $C(S)$ is the center of S) and monoids for which every element of $E(S)\setminus\{1\}$ is right zero and give a characterization when all (finitely generated, cyclic) right S-acts satisfying Conditions (PWP), (WP), (P_E) or (P) are regular. Simple monoids for which all (finitely generated, cyclic) right acts with property U (U is a

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property of acts over monoids implied by Condition (*P*)) are regular, will also be characterized.

Throughout this paper *S* will denote a monoid. We refer the reader to [3] and [5] for basic results, definitions and terminology relating to semigroups and acts over monoids and to [6], [7] for definitions and results on flatness which are used here.

A monoid *S* is called *left collapsible* if for any $p,q \in S$ there exists $r \in S$ such that $rp = rq$. A monoid *S* is called *right nil* if for every $x \in S \setminus \{1\}$, there exists $n \in N$ such that x^n is a right zero element of *S*. A monoid *S* is called *aperiodic* if for every element $x \in S$ there exists $n \in N$ such that $x^{n+1} = x^n$. A monoid *S* satisfies conditions (*K*) and (FP_2) if

(*K*): Every left collapsible submonoid of *S* contains a left zero.

(FP_2): Every left collapsible submonoid of $< E(S) >$ contains a left zero.

Let *A* be a right *S*-act. An element $a \in A$ is called *act-regular*, if there exists a homomorphism $f : aS \to S$ such that $af(a) = a$, and *A* is called a *regular act* if every $a \in A$ is an act - regular element.

A right *S*-act *A* satisfies Condition (*P*) if, for every $a, a' \in A$ and $s, s' \in S$, $as = a's'$ implies that there exist $a'' \in A$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs'$. A right *S*-act *A* satisfies Condition (*E*) if, for every $a \in A$ and $s, s' \in S$, $as = as'$ implies that there exist $a' \in A$, $u \in S$ such that $a = a'u$ and $us = us'$. A right *S*-act *A* satisfies Condition (P_E) if whenever $a, a' \in A$, $s, s' \in S$, and $as = a's'$, there exist $a'' \in A$ and $u, v, e, f \in S$ with $e^2 = e$, $f^2 = f$ such that $ae = a''ue$, $a'f = a''vf$, $es = s$, $fs' = s'$ and $us = vs'$. It is obvious that Condition (*P*) implies Condition (P_E), but the converse is not true (see [2].

We use the following abbreviations: weak kernel flatness=*WKF*; principal weak kernel flatness=*PWKF*; translation kernel flatness=*TKF*; weak homoflatness= *WP*; principal weak homoflatness = *PWP*.

Results

1. Classification of Monoids by Regularity of Acts

In this section by assuming that *U* is a property of acts over monoids implied by strong flatness, we give a characterization of monoids for which this property of their (finitely generated, cyclic) right acts implies regularity. We also give a characterization of monoids for which all finitely generated (cyclic) right acts are regular and show that, monoids for which all regular right acts are *WPF*, *WKF, PWKF, TKF* or satisfy Conditions (*P*), (*WP*) and (*PWP*) are the same as those for which all regular right acts are projective or strongly flat, which characterized in [4].

Lemma 1.1. *Let S be a monoid and A a right S-act. If for every* $a \in A$, *aS satisfies Condition (E), then A satisfies Condition (E).*

Proof. It is straightforward.□

Corollary 1.2. *Let S be a monoid and A a right S-act. If A is regular, then A satisfies Condition (E).*

Proof. Suppose that *A* is a regular right *S*-act and let $a \in A$. By [5, III, 19.3], *aS* is projective. Thus *aS* satisfies Condition (*E*) and so by Lemma 1.1, A satisfies Condition (*E*).□

Lemma 1.3. *Let S be a right PP monoid and let A be a right S-act. If A satisfies Condition (E), then for every* $a \in A$, *aS satisfies Condition (E).*

Proof. Suppose that *A* satisfies Condition (*E*) and let $as = at$ for $a \in A$ and $s, t \in S$. Then there exist $a' \in A$ and $u \in S$ such that $a = a'u$ and $us = ut$. Since *S* is right *PP*, then there exists $e \in E(S)$ such that $\ker \lambda_e = \ker \lambda_u$. Thus $es = et$, $u = ue$, and so $a = ae$. Hence *aS* satisfies Condition (*E*) as required.□

Theorem 1.4. *Let S be a monoid and let U be a property of S-acts implied by strong flatness. Then:*

(1) *All right S-acts having property U are regular if and only if S is right PP, satisfies Condition (K) and every right S-act having property U satisfies Condition (E).*

(2) *All finitely generated right S-acts having property U are regular if and only if S is right PP, satisfies Condition (K) and every finitely generated right S-act having property U satisfies Condition (E).*

(3) *All cyclic right S-acts having property U are regular if and only if S is right PP, satisfies Condition (K) and every cyclic right S-act having property U satisfies Condition (E).*

Proof. (1) Suppose that all right *S*-acts having property *U* are regular. Then by Corollary 1.2, every right *S*-act having property *U* satisfies Condition (*E*). Since S_S is strongly flat, then by assumption S_S is regular. Thus by [5, III, 19.3], all principal right ideals of *S* are projective, and so by [5, IV, 11.15], *S* is right *PP*. Also by assumption and [5, III, 19.3], all strongly flat cyclic right *S*-acts are projective. Thus by [5, IV, 11.2], *S* satisfies Condition (*K*).

Conversely, suppose that *A* is a right *S*-act with property *U*. Since by assumption *A* satisfies Condition

(E) and S is right *PP*, then by Lemma 1.3, for every $a \in A$, aS satisfies Condition (E) and so aS is strongly flat. Since S satisfies Condition (K), then by [5, IV, 11.2], aS is projective and so by [5, III, 19.3], A is regular.

The proofs of 2 and 3 are similar.□

Now, from Theorem 1.4, we have

Corollary 1.5. *For any monoid S, the following statements are equivalent:*

(1) *All strongly flat right S-acts are regular.*

(2) *All strongly flat finitely generated right S-acts are regular.*

(3) *All strongly flat cyclic right S-acts are regular.*

(4) *S is right PP and satisfies Condition (K).*

Corollary 1.6. *For any monoid S, the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (E) are regular.*

(2) *All finitely generated right S-acts satisfying Condition (E) are regular.*

(3) *All cyclic right S-acts satisfying Condition (E) are regular.*

(4) *S is right PP and satisfies Condition (K).*

Corollary 1.7. *For any monoid S, the following statements are equivalent:*

(1) *All equalizer flat right S-acts are regular.*

(2) *All equalizer flat finitely generated right S-acts are regular.*

(3) *All equalizer flat cyclic right S-acts are regular.*

(4) *S is right PP and satisfies Condition (K).*

Theorem 1.8. *For any monoid S, the following statements are equivalent:*

(1) *All torsion free right S-acts are regular.*

(2) *All torsion free finitely generated right S-acts are regular.*

(3) *All torsion free cyclic right S-acts are regular.*

(4) *All torsion free monocyclic right S-acts are regular.*

(5) $S = \{0,1\}$ *or* $S = \{1\}$.

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious. $(4) \Rightarrow (5)$. By assumption and [5, III, 19.3], all torsion free monocyclic right S-acts are projective, and so by [5, IV, 11.14], $S = \{0,1\}$ *or* $S = \{1\}$. $(5) \Rightarrow (1)$. By [5, IV, 11.14], all cyclic right S-acts are projective, and so by [5, III, 19.3], all right S-acts are regular.□

Lemma 1.9. *Let S be a monoid. If S is left collapsible and every element of* $S \setminus \{1\}$ *is right zero, then* $S = \{0,1\}$ *or* $S = \{1\}$.

Proof. Suppose that S is left collapsible and every element of $S \setminus \{1\}$ right zero. We claim that $|S| \leq 2$. Otherwise, there exist $e, f \in S \setminus \{1\}$ such that $e \neq f$. Since S is left collapsible, then there exists $u \in S$ such that $ue = uf$. But e and f are right zero, and so $e = f$, which is a contradiction. Thus $|S| \leq 2$, and so either $S = \{0,1\}$ *or* $S = \{1\}$ as required.□

Corollary 1.10. *Let S be a monoid. If every element of* $S \setminus \{1\}$ *is right zero, then S satisfies Condition (K).*

Proof. It suffices to show that every left collapsible submonoid of S contains a left zero, but this is true by Lemma 1.9.□

Lemma 1.11. *Let S be a monoid. If S is right PP and right nil, then every element of* $S \setminus \{1\}$ *is right zero.*

Proof. Suppose that S is a right *PP* and right nil monoid and let $1 \neq s \in S$. Then there exists the smallest positive integer n such that s^n is right zero. Thus $s^{n+1} = s^n$ and so $ss^n = ss^{n-1}$ implies that $(s^n, s^{n-1}) \in \ker \lambda_s$. Since S is right *PP*, there exists $e \in E(S)$ such that $\ker \lambda_s = \ker \lambda_e$. Thus $es^n = es^{n-1}$ and $se = s$. Since n is the smallest positive integer such that s^n is right zero, then $e \neq 1$ and since S is right nil, e is right zero, and so $s = se = e$ is right zero as required.□

Theorem 1.12. *For any monoid S, the following statements are equivalent:*

(1) *All principally weakly flat right S-acts are regular.*

(2) *All principally weakly flat finitely generated right S-acts are regular.*

(3) *All principally weakly flat cyclic right S-acts are regular.*

(4) $S = \{0,1\}$ *or* $S = \{1\}$.

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious. $(3) \Rightarrow (4)$. By Theorem 1.4, S is right *PP*, satisfies Condition (K) and every principally weakly flat cyclic right S-act satisfies Condition (E). Thus every weakly flat cyclic right S-act satisfies Condition (E), and so every weakly flat cyclic right S-act is strongly flat. Hence by [5, IV,11.10], S is right nil and so by Lemma 1.11, every element of $S \setminus \{1\}$ is right zero. By [5, III, 10.2], Θ_S is principally weakly flat, and so by assumption Θ_S ($\Theta_S = \{\theta\}$) is the one-element right S-act) satisfies Condition (E). Thus by [5, III, 14.3], S is left collapsible and hence by Lemma 1.9, either $S = \{0,1\}$ *or* $S = \{1\}$.

$(3) \Rightarrow (4)$. By Theorem 1.8, it is obvious.□

Lemma 1.13. *If every element of* $S \setminus \{1\}$ *is right zero, then all weakly flat right S-acts satisfy Condition (E).*

Proof. Let A be a weakly flat right S-act and suppose that $as = as'$ for $a \in A$ and $s, s' \in S$. Since every element of $S \setminus \{1\}$ is right zero, then S is a regular monoid, and so it is left *PP*. Thus by [5, III, 11.9], there exist $u, v \in S$, $e, f \in E(S)$ and $a'' \in A$ such that $ae = a''u$, $af = a''v$, $es = s$, $fs' = s'$ and $us = vs'$.

If $s \neq 1$ and $s' \neq 1$, then $s = s'$. Thus $a = a.1$ and $1.s = 1.s'$.

If at least one of s or s' is equal to 1, for example if $s' = 1$, then $a = as$ and $s.s = s.1$.

Thus A satisfies condition (E) as required.□

Theorem 1.14. *For any monoid S, the following statements are equivalent:*

(1) *All weakly flat right S-acts are regular.*

(2) *All weakly flat finitely generated right S-acts are regular.*

(3) *All weakly flat cyclic right S-acts are regular.*

(4) *Every element of* $S \setminus \{1\}$ *is right zero.*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious.

$(3) \Rightarrow (4)$. By Theorem 1.4, S is right *PP*, satisfies Condition (K) and every weakly flat cyclic right S-act satisfies Condition (E). Thus every weakly flat cyclic right S-act is strongly flat. Hence by [5, IV, 11.10], S is right nil, and so by Lemma 1.11, every element of $S \setminus \{1\}$ is right zero as required.

$(4) \Rightarrow (1)$. By Lemma 1.13, all weakly flat right S-acts satisfy Condition (E). Since every element of $S \setminus \{1\}$ is right zero, then S is regular, and so S is right *PP*. Also by Corollary 1.10, S satisfies Condition (K). Thus by Theorem 1.4, all weakly flat right S-acts are regular.□

Theorem 1.15. *For any monoid S, the following statements are equivalent:*

(1) *All flat right S-acts are regular.*

(2) *All flat finitely generated right S-acts are regular.*

(3) *All flat cyclic right S-acts are regular.*

(4) *Every element of* $S \setminus \{1\}$ *is right zero.*

Proof. It is similar to the proof of Theorem 1.14.□

Note that by Theorem 1.4, all right S-acts satisfying Condition (P) are regular if and only if S is right *PP*, satisfies Condition (K) and every right S-act satisfying Condition (P) is strongly flat. Thus, if we know monoids for which all right acts satisfying Condition (P) are strongly flat, then we can give a characterization of monoids for which all right acts satisfying Condition (P) are regular. For finitely generated (cyclic) acts we have the following:

Theorem 1.16. *For any monoid S, the following statements are equivalent:*

(1) *All finitely generated right S-acts satisfying Condition (P) are regular.*

(2) *All cyclic right S-acts satisfying Condition (P) are regular.*

(3) *S is aperiodic, right PP and satisfies Condition (K).*

Proof. Implication $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$. By Theorem 1.4, S is right *PP*, satisfies Condition (K) and all cyclic right S-acts satisfying Condition (P) satisfy Condition (E). Thus all cyclic right S-acts satisfying Condition (P) are strongly flat, and so by [5, IV,10.2], S is aperiodic.

$(3) \Rightarrow (1)$. By [5, IV, 10.2], all finitely generated right S-acts satisfying Condition (P) are strongly flat, and so all finitely generated right S-acts satisfying Condition (P) satisfy Condition (E). Thus by Theorem 1.4, all finitely generated right S-acts satisfying Condition (P) are regular.□

Now, we show that monoids for which all finitely generated (cyclic, monocyclic) right acts are regular are the same as those for which all right acts are regular, characterized by Kilp and Knauer.

Theorem 1.17. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts are regular.*

(2) *All finitely generated right S-acts are regular.*

(3) *All cyclic right S-acts are regular.*

(4) *All monocyclic right S-acts are regular.*

(5) $S = \{0,1\}$ *or* $S = \{1\}$.

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (5)$. By Theorem 1.8, it is obvious.

$(5) \Rightarrow (1)$. By [5, IV, 11.14], all cyclic right S-acts are projective, and so by [5, III, 19.3], all right S-acts are regular.□

Characterization of monoids over which all regular right S-acts are projective or strongly flat were considered by Kilp and Knauer in [4] and characterization of monoids over which all regular right S-acts are (weakly) flat were considered by Liu in [8]. Now, we consider monoids for which regularity of acts implies weak pullback flatness, homoflatness, kernel flatness or satisfy Condition (P).

Theorem 1.18. *Let S be a monoid and suppose that there exists at least a regular right S-act. Then the following statements are equivalent:*

(1) *All regular right S-acts are weakly pullback flat.*
(2) *All regular right S-acts satisfy Condition (P).*
(3) *All regular right S-acts satisfy Condition (WP).*
(4) *All regular right S-acts satisfy Condition (PWP).*
(5) *All regular right S-acts are WKF.*
(6) *All regular right S-acts are PWKF.*
(7) *All regular right S-acts are TKF.*
(8) *Every idempotent of T generates a minimal right ideal, where T is the largest regular right ideal of S.*

Proof. Implications $(1)\Rightarrow(2)\Rightarrow(3)\Rightarrow(4)$ and $(1)\Rightarrow(5)\Rightarrow(6)\Rightarrow(7)\Rightarrow(4)$ are obvious.

$(4)\Rightarrow(8)$. By assumption there exists a regular right *S*-act, and so there exists the largest regular right ideal *T* of *S*. It is obvious that *eS* is regular for $e\in T$. Now suppose that K_S is a right ideal of *S* such that $K_S\subset eS$. If $A_S=eS\coprod^{K_S}eS$, then $A_S=(e,x)S\dot{\bigcup}K_S\dot{\bigcup}(e,y)S$ where $(e,x)S\dot{\bigcup}K_S$ and $(e,y)S\dot{\bigcup}K_S$ are subacts of A_S isomorphic to *eS*. Thus A_S is regular and so by assumption A_S satisfies Condition (*PWP*). If for $w\in S$, $ew\in K_S$, then *ew* = (*e, x*)*ew* = (*e, y*)*ew* and so there exist $a\in A_S$ and $u,v\in S$ such that (*e, x*) = *au*, (*e, y*) = *av* and *uew* = *vew*. Thus there exist $et\in eS\setminus K_S$ and $et'\in eS\setminus K_S$ such that *a* = (*et, x*) and *a* =(*et', y*), which is a contradiction. Thus A_S does not satisfy Condition (*PWP*), and so we have a contradiction. Thus for every $e\in E(T)$, *eS* is a minimal right ideal of *S*.

$(8)\Rightarrow(1)$. It is true by [4, Theorem 3.9].□

2. Classification of Monoids S with E(S)⊆C(S) by Regularity of Acts

As we mentioned, up to now there is no characterization for monoids over which all right *S*-acts satisfying Conditions (*P*) are regular. But in this section we give a characterization of monoids *S* with $E(S)\subseteq C(S)$ for which all (finitely generated, cyclic) right *S*-acts satisfying Conditions (*P*), (*PWP*) or (P_E) are regular.

Lemma 2.1. *Let S be a right PP and aperiodic monoid. If* $E(S)\subseteq C(S)$ *or every element of* $E(S)\setminus\{1\}$ *is right zero, then* $E(S)=S$.

Proof. If $S\neq E(S)$, then there exists $s\in S\setminus E(S)$. Since *S* is aperiodic, there exists the smallest positive integer *n* such that $n\geq 2$ and $s^n=s^{n+1}$. Since *S* is right *PP*, there exists $e\in E(S)$ such that $\ker\lambda_s=\ker\lambda_e$. Now $ss^{n-1}=ss^n$ implies that $(s^{n-1},s^n)\in\ker\lambda_s=\ker\lambda_e$. Thus $se=s$ and $es^{n-1}=es^n$. Since *n* is the smallest positive integer such that $s^n=s^{n+1}$, then $e\neq 1$, also $se=s$ implies that $s^ne=s^n$ and $s^{n-1}e=s^{n-1}$. If $E(S)\subseteq C(S)$, then we have $s^n=s^{n-1}$, which is a contradiction.

If every element of $E(S)\setminus\{1\}$ is right zero, then $s=se=e$, which is also a contradiction. Thus $E(S)=S$ as required.□

Theorem 2.2. *Let S be a monoid with* $E(S)\subseteq C(S)$ *and suppose that U is a property of S-acts implied by property (P). If all cyclic right S-acts having property U are regular, then S is an idempotent monoid and satisfies Condition* (FP_2).

Proof. If all cyclic right *S*-acts having property *U* are regular, then by Theorem 1.4, *S* is right *PP*, satisfies Condition (*K*) and every cyclic right *S*-act having property *U* satisfies Condition (*E*). Since Condition (*P*) implies property *U*, then all cyclic right *S*-acts satisfying Condition (*P*) are strongly flat, and so by [5, IV, 10.2], *S* is aperiodic. Thus by Lemma 2.1, $E(S)=S$, that is, *S* is an idempotent monoid, and so Conditions (*K*) and (FP_2) coincide.□

Notice that commutative monoids and monoids *S* with $E(S)=\{1\}$ are those for which $E(S)\subseteq C(S)$, and so satisfy Theorem 2.2.

Corollary 2.3. *For any monoid S with* $E(S)\subseteq C(S)$, *the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (PWP) are regular.*
(2) *All finitely generated right S-acts satisfying Condition (PWP) are regular.*
(3) *All cyclic right S-acts satisfying Condition (PWP) are regular.*
(4) $S=\{0,1\}$ *or* $S=\{1\}$.

Proof. Implications $(1)\Rightarrow(2)\Rightarrow(3)$ are obvious.

$(3)\Rightarrow(4)$. By Theorem 2.2, *S* is an idempotent monoid. Also by Theorem 1.4, all cyclic right *S*-acts satisfying Condition (*PWP*) satisfy Condition (*E*). Thus all cyclic right *S*-acts satisfying Condition (*PWP*) are strongly flat, and so by [6, Corollary 3.58], $S=\{0,1\}$ *or* $S=\{1\}$.

$(4)\Rightarrow(1)$. It is obvious by Theorem 1.17.□

Corollary 2.4. *For any monoid S with* $E(S)\subseteq C(S)$, *the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (P_E) are regular.*

(2) *All finitely generated right S-acts satisfying Condition (P_E) are regular.*

(3) *All cyclic right S-acts satisfying Condition (P_E) are regular.*

(4) $S = \{0,1\}$ *or* $S = \{1\}$.

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious.

$(3) \Rightarrow (4)$. By Theorem 2.2, S is an idempotent monoid. Thus *S* is left *PP*, and so by [2, Theorem 2.5], Condition (P_E) and weak flatness coincide. Thus by Theorem 1.14, every element of $S \setminus \{1\}$ is right zero, and so *S* is a commutative. Thus every element different from 1 is zero, and so $S = \{0,1\}$ or $S = \{1\}$.

$(4) \Rightarrow (1)$. By Theorem 1.17, it is obvious.□

Corollary 2.5. *Let S be a monoid with* $E(S) \subseteq C(S)$. *Then the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (P) are regular.*

(2) *All finitely generated right S-acts satisfying Condition (P) are regular.*

(3) *All cyclic right S-acts satisfying Condition (P) are regular.*

(4) *S is an idempotent monoid and satisfies Condition (FP_2).*

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious.

$(3) \Rightarrow (4)$. If all cyclic right *S*-acts satisfying Condition (*P*) are regular, then by Theorem 2.2, *S* is an idempotent monoid and satisfies Condition (FP_2).

$(4) \Rightarrow (1)$. Suppose that *S* is an idempotent monoid and satisfies Condition (FP_2). Then *S* is right *PP*, Conditions (*K*) and (FP_2) coincide and by [1, Proposition, 2.13], pullback flatness and Condition (*P*) also coincide. Thus Condition (*P*) implies Condition (*E*), and so by Theorem 1.4, all right *S*-acts satisfying Condition (*P*) are regular.□

3. Classification of Monoids S with Elements of E(S)\{1} Right Zero by Regularity of Acts

In this section we characterize monoids S with elements of $E(S) \setminus \{1\}$ right zero such that all (finitely generated, cyclic) right *S*-acts satisfying Conditions (*PWP*), (*WP*), (P_E) or (*P*) are regular.

Theorem 3.1. *Let S be a monoid with elements of* $E(S) \setminus \{1\}$ *right zero and let U be a property of S-acts which is implied by Condition (P). Then:*

(1) *If all right S-acts having property U are regular, then every element of* $S \setminus \{1\}$ *is right zero. Moreover, if U implies weak flatness, then the converse is also true.*

(2) *If all finitely generated right S-acts having property U are regular, then every element of* $S \setminus \{1\}$ *is right zero. Moreover, if U implies weak flatness, then the converse is also true.*

(3) *If all cyclic right S-acts having property U are regular, then every element of* $S \setminus \{1\}$ *is right zero. Moreover if U implies weak flatness, then the converse is also true.*

Proof. (1) If all right *S*-acts having property *U* are regular, then by Theorem 1.4, *S* is right *PP*, satisfies Condition (*K*) and every right *S*-act having property *U* satisfies Condition (*E*). Since Condition (*P*) implies property *U*, then all right *S*-acts satisfying Condition (*P*) are strongly flat. Thus all cyclic right *S*-acts satisfying Condition (*P*) are strongly flat, and so by [5, IV, 10.2], *S* is aperiodic. Hence by Lemma 2.1, *S* = *E*(*S*), and so every element of $S \setminus \{1\}$ is right zero as required.

If *U* implies weak flatness, then the converse of (1) is also true by Theorem 1.14.

Parts 2 and 3 of theorem can easily be proved by using parts 2 and 3 of Theorem 1.4.□

Now, from Theorems 1.17, 3.1, and [6, Corollary 3.58] we have,

Corollary 3.2. *For any monoid S with elements of* $E(S) \setminus \{1\}$ *right zero, the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (PWP) are regular.*

(2) *All finitely generated right S-acts satisfying Condition (PWP) are regular.*

(3) *All cyclic right S-acts satisfying Condition (PWP) are regular.*

(4) $S = \{0,1\}$ *or* $S = \{1\}$.

Since $(P) \Rightarrow (WP) \Rightarrow (WF)$ then by Theorem 3.1, we have

Corollary 3.3. *For any monoid S with elements of* $E(S) \setminus \{1\}$ *right zero, the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (WP) are regular.*

(2) *All finitely generated right S-acts satisfying Condition (WP) are regular.*

(3) *All cyclic right S-acts satisfying Condition (WP) are regular.*

(4) *Every element of* $S \setminus \{1\}$ *is right zero.*

Corollary 3.4. *For any monoid S with elements of*

$E(S)\setminus\{1\}$ *right zero, the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (P) are regular.*

(2) *All finitely generated right S-acts satisfying Condition (P) are regular.*

(3) *All cyclic right S-acts satisfying Condition (P) are regular.*

(4) *Every element of* $S\setminus\{1\}$ *is right zero.*

Note that since $(P)\Rightarrow(P_E)\Rightarrow(WF)$, then we have the same results as in Corollary 3.4, when (*P*) is replaced by (P_E).

4. Classification of Simple Monoids by Regularity of Acts

In this section by assuming that *U* is a property of acts over monoids implied by Condition (*P*), we give a classification of monoids for which this property of their (finitely generated, cyclic) right acts implies regularity.

Theorem 4.1. *Let S be a simple monoid and let U be a property of S-acts. If Condition (P) of S-acts implies U, then the following statements are equivalent:*

(1) *All right S-acts having property U are regular.*

(2) *All finitely generated right S-acts having property U are regular.*

(3) *All cyclic right S-acts having property U are regular.*

(4) $S=\{1\}$.

Proof. Implications $(1)\Rightarrow(2)\Rightarrow(3)$ are obvious.

$(3)\Rightarrow(4)$. Suppose that all cyclic right *S*-acts having property *U* are regular. Then all cyclic right *S*-acts satisfying condition (*P*) are regular. Thus by (3) of Theorem 1.4, all cyclic right *S*-acts satisfying condition (*P*) are strongly flat, and so by [5, IV, 10.2], S is aperiodic. On the other hand if *A* is a cyclic right *S*-act with property *U*, then *A* is regular, and so by [5, III, 19.3], for every $a\in A$, *aS* is projective. Thus by [5, I, 5.8], and [5, III, 17.7], there exists $e\in E(S)$ such that $aS\cong eS$. But by [5, IV,12.2], *eS* is generator, and so *aS* is projective generator. Since S_S satisfies Condition (*P*) and by assumption Condition (*P*) implies property *U*, then for every $s\in S$, *sS* is projective generator, and so by [5, III, 18.8], there exists $f\in E(S)$ such that $sS\cong fS$ and $f\ \mathcal{J}\ 1$. Since *S* is aperiodic, then *S* is periodic and so by [5, I, 3.26], $f=1$. Thus $sS\cong S$, and so *sS* is free. Hence by [5, I,5.20], and [5, I, 3.26], $\ker\lambda_s\le\ker\lambda_1$, and so for all $s,t_1,t_2\in S$, $st_1=st_2$ implies that $t_1=t_2$, that is, *S* is left cancellative. Since *S* is aperiodic, then $S=\{1\}$ as required.

$(4)\Rightarrow(1)$. By Theorem 1.17, it is obvious.□

Note that in Theorem 4.1, *U* can be replaced by flat, (principally) weakly flat, torsion free and Conditions (*P*), (P_E), (*WP*) and (*PWP*).

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