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Asymptotic Behavior of Weighted Sums of Weakly Negative Dependent Random Variables

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Abstract

Let $\{X_j; j \geq 1\}$ be a sequence of weakly negative dependent (denoted by, WND) random variables with common distribution function F and let $\{\theta_j; j \geq 1\}$ be other sequence of positive random variables independent of $\{X_j; j \geq 1\}$ and $P[a \leq \theta_j \leq b] = 1$ for some $0 < a \leq b < \infty$ and for all $j \geq 1$. In this paper, we study the asymptotic behavior of the tail probabilities of the maximum, weighted sums, randomly weighted sums and randomly indexed weighted sums of heavy-tailed weakly negative dependent random variables, say, $\max_{1\leq j\leq n} X_j$, $\sum_{j=1}^{n} c_j X_j$, $\sum_{j=1}^{n} \theta_j X_j$, and $\sum_{j=1}^{N} \theta_j X_j$, respectively, where $\{c_j; 1 \leq j \leq n\}$ are n bounded positive real numbers and N is a nonnegative integer-valued random variables, independent of θ_i and X_i for all $i \geq 1$. In fact, for a large class of heavy-tailed distribution functions, we show that the asymptotic relations,

$$P[\max_{1\leq j\leq n} \theta_j X_j > x] \sim P[\sum_{j=1}^{n} \theta_j X_j > x] \sim \sum_{j=1}^{n} P[\theta_j X_j > x],$$

hold as $x \to \infty$. Finally, if $E(N) < \infty$ and also $\{\theta_j; j \geq 1\}$ is a sequence of identical independent positive random variables, then we prove that $P[\sum_{j=1}^{N} \theta_j X_j > x] \sim E(N).P[\theta_1 X_1 > x], \quad as \quad x \to \infty.$

Keywords: Weakly negative dependent; Heavy-tailed; Asymptotic behavior; Weighted sums

Introduction

Many authors have extensively investigated the asymptotic behaviors of the tail probability of partial sums and weighted sums of independent heavy-tailed random variables. In recent years, Cai and Tang [2]

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generalized the well-known max-sum equivalence and convolution closure of two random variables when they are independent and in follow of them Kiaw and Tang [11], Chen *et al.* [3] and Gluk *et al.* [9], have been provided asymptotic behavior of weighted sums of subexponential random variables and Large deviations of weighted sums and partial sums for negatively dependent (ND) random variables have been extended by Chen *et al.* [4], Tang [15] and for negatively associated (NA) ones, it is provided by Wang and Tang [16] and Lio [12]. In this paper we study the asymptotic behaviors of the tail probabilities, $P(\sum_{i=1}^{n} c_i X_i > x)$, $P(\sum_{i=1}^{N} \theta_i X_i > x)$ and $P[\max_{1\le j\le n} \theta_j X_j > x]$ as x tends to infinity; where, $\{c_j; 1 \le j \le n\}$ are n bounded positive real numbers and $\{\theta_j; j \ge 1\}$ is a sequence of positive random variables which are independent of the sequence $\{X_j; j \ge 1\}$, for a class of heavy-tailed distribution functions. In fact, for a large class of heavy-tailed distribution functions, we show that the asymptotic relations,

$$P[\max_{1\le j\le n} \theta_j X_j > x] \sim P[\sum_{j=1}^{n} \theta_j X_j > x]$$

$$\sim \sum_{j=1}^{n} P[\theta_j X_j > x],$$

hold as $x \to \infty$. In addition, if $E(N) < \infty$ and also $\{\theta_j; j \ge 1\}$ is a sequence of identical independent positive random variables, then we prove that

$$P[\sum_{j=1}^{N} \theta_j X_j > x] \sim E(N).P[\theta_1 X_1 > x], \textit{ as } x \to \infty.$$

Throughout this paper, all distribution functions will be defined on $[0,\infty)$ and $f(x) \sim g(x)$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$. We denote the tail of distribution of F by $\bar{F}(x) = 1 - F(x)$, convolution of distributions F and G by $F*G$ and denote n^{th} convolution of F by $F^{(n)}$ and $\bar{F}^{(n)} = 1 - F^{(n)}$. In the following, some well known classes of heavy-tailed distribution functions are listed.

1. Dominated variation (D): The distribution function F belongs to D if for any $0 < t < 1$,

$$\limsup_{x\to\infty} \frac{\bar{F}(tx)}{\bar{F}(x)} < \infty .$$

2. Consistency varying tailed (C): The distribution function F belongs to C if,

$$\lim_{t\uparrow 1} \limsup_{x\to\infty} \frac{\bar{F}(tx)}{\bar{F}(x)} = 1 \text{ or } \lim_{t\downarrow 1} \limsup_{x\to\infty} \frac{\bar{F}(tx)}{\bar{F}(x)} = 1 .$$

3. Long-tailed (L): The distribution function F belongs to L if for any $t > 0$,

$$\lim_{x\to\infty} \frac{\bar{F}(x-t)}{\bar{F}(x)} = 1 .$$

For more details about the heavy-tailed distribution functions, see [6], [1], and [10]. Moreover, as other classes of heavy-tailed distributions, we consider the following relation holds for the distribution functions,

$$\int_0^{\infty} \bar{F}(u)du < \infty. \tag{1}$$

It is easy to find some evidence for the relation (1) like the distribution functions that belong to the class of subexponential distribution functions that were defined in [5].

Definition 1. The random variables $X_1,\ldots,X_n$ are said to be weekly negatively dependent (WND) if for each n and all $x_1,\ldots,x_n$, there exist some $C \ge 1$ such that

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) \le C.\prod_{i=1}^{n} f_{X_i}(x_i) \tag{2}$$

The class of WND random variables is well defined and a large class of these random variables can be found. Some of them are presented in the following example.

Example *i***)** Suppose that the random vector (X,Y) have joint half-normal distribution, then

$$f(x,y) = \frac{2}{\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{x^2 + y^2 - 2\rho xy\right\}\right]; \quad x,y > 0,$$

$$f_i(x) = \sqrt{\frac{2}{\pi}} \exp\left[-\frac{1}{2}x^2\right]; \; i = 1,2.$$

If $-1 < \rho \le 0$, then X and Y are ND random variables (see [7]). Moreover,

$$\frac{f(x,y)}{f_1(x)f_2(y)}$$

$$= \frac{1}{\sqrt{1-\rho^2}} \exp\left\{ -\frac{\rho^2}{2(1-\rho^2)}(x^2+y^2) + \frac{\rho}{1-\rho^2}xy \right\}$$

$$\leq \frac{1}{\sqrt{1-\rho^2}}.$$

So, X and Y are WND with $C = 1/\sqrt{1-\rho^2}$.

ii) Let (X, Y) be two random variables with joint distribution function $F(x, y)$ belongs to family of bivariate distributions of Farlie-Gumbel-Morgenstern (FGM). Then the joint density function of (X, Y) is as follows,

$$f(x, y) = f_1(x) f_2(y)[1 + \alpha(1-2F_1(x))(1-2F_2(y))];$$

$$x, y \in R, \ -1 \leq \alpha \leq 1.$$

Where F_1 and F_2 are marginal distributions of X and Y respectively. (For more details see [13]). Therefore, the random variables X and Y are WND with $C = 1+|\alpha|$. Moreover, we know if $-1 < \alpha < 0$, then X and Y are ND. (see [7]).

In particular if $F_i(x) = 1-(\alpha/\alpha + x)^\beta, x > 0, i = 1,2$ ($X \sim Pareto(\alpha, \beta)$ where $\alpha > 0$ and β is positive integer), then $F_i \in C \subset D \bigcap L$. For these examples it is easy to see that the condition (1) holds.

The following Lemmas are important technical tools in the proof of our results. Since the proofs of these Lemmas are easy, we omitted them.

Lemma 1. Let $X_1,\ldots,X_n$ be WND random variables with joint distribution function $F(x_1,\ldots,x_n)$ and marginal distributions $F_1(x_1),\ldots,F_n(x_n)$, respectively. If $h_1(.),\ldots,h_n(.)$ are monotone measurable functions, then

i. The random variables $h_1(X_1),\ldots,h_n(X_n)$ are WND.

ii. For all $x_1,\ldots,x_n \in R$,

$$F(x_1,x_2,\ldots,x_n) \leq C.\prod_{i=1}^{n} F_i(x_i)$$

and

$$\overline{F}(x_1,x_2,\ldots,x_n) \leq C.\prod_{i=1}^{n} \overline{F}_i(x_i).$$

Lemma 2. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive real numbers, then

$$\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \leq \max\left\{ \frac{a_i}{b_i}; i = 1,\ldots,n \right\}.$$

Lemma 3. Let X be a random variable with distribution function F and cX with distribution function F' , where c is a positive constant, then

i. $F \in L$ if and only if $F' \in L$

ii. $F \in D$ if and only if $F' \in D$

Results

In this section, first we present the following Lemma and Theorems to be used in the proof of our main results. Then, we derive the asymptotic behaviors of the tail probabilities of weighted sums of WND random variables. In fact, we prove some equivalence statements for tail probabilities.

Lemma 4. Let X_1 and X_2 be two WND random variables with common distribution function $F \in L$, if F satisfies in condition (1), then for all $x > 2\upsilon > 0$,

$$\lim_{\upsilon\to\infty}\lim_{x\to\infty}\int_{\upsilon}^{x}\frac{\overline{F}(x-u)}{\overline{F}(x)}dF(u) = 0.$$

Proof. We can write

$$\int_{\upsilon}^{x}\frac{\overline{F}(x-u)}{\overline{F}(x)}dF(u) = \int_{\upsilon}^{x-\upsilon}\frac{\overline{F}(x-u)}{\overline{F}(x)}dF(u) + \int_{x-\upsilon}^{x}\frac{\overline{F}(x-u)}{\overline{F}(x)}dF(u) \quad (3)$$

$$\leq \int_{\upsilon}^{x-\upsilon}\frac{\overline{F}(x-u)}{\overline{F}(x)}dF(u) + \overline{F}(x-\upsilon),$$

and , for any $x > 2\upsilon$, we get

$$\int_{\upsilon}^{x-\upsilon}\overline{F}(x-u)dF(u) = \int_{\upsilon}^{x-\upsilon}\overline{F}(u)d\overline{F}(x-u)$$

$$\leq \int_{\upsilon}^{x+\upsilon}\overline{F}(u)d\overline{F}(x-u) = I_1.$$

Now, let

$$I = \int_{0}^{\infty}\overline{F}(u)du = \sum_{n=0}^{\infty}\int_{nh}^{(n+1)h}\overline{F}(u)du.$$

We have,

$$h\sum_{n=1}^{\infty}\bar{F}(nh)=h\sum_{n=0}^{\infty}\bar{F}(nh+h)\le I$$
$$\le h\sum_{n=0}^{\infty}\bar{F}(nh)<\infty. \quad (4)$$

Since $\bar{F}(nh+h+\upsilon)\le\bar{F}(u)\le\bar{F}(nh+\upsilon)$ for $u\in[nh+\upsilon, nh+h+\upsilon]$, we can write,

$$\sum_{n=0}^{N_0-1}\bar{F}[(n+1)h+\upsilon]\left\{\bar{F}(x-(n+1)h-\upsilon)-\bar{F}(x-nh-\upsilon)\right\}$$
$$=\sum_{n=0}^{N_0-1}\int_{nh+\upsilon}^{(n+1)h+\upsilon}\bar{F}[(n+1)h+\upsilon]d\bar{F}(x-u)$$
$$\le\sum_{n=0}^{N_0-1}\int_{nh+\upsilon}^{(n+1)h+\upsilon}\bar{F}(u)d\bar{F}(x-u)=I_1$$
$$\le\sum_{n=0}^{N_0-1}\int_{nh+\upsilon}^{(n+1)h+\upsilon}\bar{F}[nh+\upsilon]d\bar{F}(x-u)$$
$$=\sum_{n=0}^{N_0-1}\bar{F}[nh+\upsilon]\left\{\bar{F}(x-(n+1)h-\upsilon)-\bar{F}(x-nh-\upsilon)\right\},$$

where $N_0=[x/h]$. Then we get

$$\frac{I_1}{\bar{F}(x)}\le$$
$$\sum_{n=0}^{N_0-1}\bar{F}[nh+\upsilon]\left\{\frac{\bar{F}(x-(n+1)h-\upsilon)}{\bar{F}(x)}-\frac{\bar{F}(x-nh-\upsilon)}{\bar{F}(x)}\right\}.$$

Since $F\in L$, then when x tends to infinity, for all values of n and υ,

$$\left\{\frac{\bar{F}(x-(n+1)h-\upsilon)}{\bar{F}(x)}-\frac{\bar{F}(x-nh-\upsilon)}{\bar{F}(x)}\right\}$$

tends to zero. Therefore for sufficient large x and for any $\varepsilon>0$, we have,

$$K_2\le\frac{I_1}{\bar{F}(x)}\le\varepsilon\sum_{n=0}^{N_0-1}\bar{F}[nh+\upsilon]<\varepsilon\sum_{n=0}^{\infty}\bar{F}(nh)<\varepsilon.M.$$

The final inequality is valid by (4). This and (3) complete the proof. ■

Theorem 1. Suppose that X_1 and X_2 are two WND random variables with the distribution functions F_1 and F_2, respectively, where $F_i\in L\,;i=1,2$. If F_1 satisfies in condition (1) and $\sup_x\bar{F}_2(x)/\bar{F}_1(x)<\infty$, then

$$P(X_1+X_2>x)\sim P(X_1>x)+P(X_2>x)\ \ as\ x\to\infty.$$

Proof. By assumption of $\sup_x\bar{F}_2(x)/\bar{F}_1(x)\le K<\infty$ and Lemma 1, for every $\upsilon>0$, we get

$$\bar{H}(x)=P(X_1+X_2>x\,;X_1<\upsilon)$$
$$+P(X_1+X_2>x\,;\upsilon<X_1<x)+P(X_1>x)$$
$$\le C\int_0^{\upsilon}\bar{F}_2(x-u)dF_1(u)+C\int_{\upsilon}^{x}\bar{F}_2(x-u)dF_1(u)+\bar{F}_1(x)$$
$$\le C\bar{F}_2(x-\upsilon)+\bar{F}_1(x)\times CK\int_{\upsilon}^{x}\frac{\bar{F}_1(x-u)}{\bar{F}_1(x)}dF_1(u)+\bar{F}_1(x)$$
$$\le\left\{\bar{F}_1(x)+\bar{F}_2(x)\right\}$$
$$\times\left\{\left[1+CK\int_{\upsilon}^{x}\frac{\bar{F}_1(x-u)}{\bar{F}_1(x)}dF_1(u)\right]\vee\left[\frac{C\bar{F}_2(x-\upsilon)}{\bar{F}_2(x)}\right]\right\},$$

where $x\vee y=\max\{x,y\}$. So

$$\lim_{\upsilon\to\infty}\limsup_{x\to\infty}\frac{\bar{H}(x)}{\bar{F}_1(x)+\bar{F}_2(x)}\le 1. \quad (5)$$

Since $F_2\in L$ and F_1 satisfies in the condition (1), hence Lemma 4 implies (5). Moreover, for any $x>0$ we have,

$$P(X_1+X_2>x)=P(X_1>x)+P(X_2>x)$$
$$+P(X_1+X_2>x\,;X_1<x\,;X_2<x)-P(X_1>x\,;X_2>x)$$
$$\ge P(X_1>x)+P(X_2>x)-P(X_1>x\,;X_2>x)$$
$$\ge P(X_1>x)+P(X_2>x)-CP(X_1>x)P(X_2>x).$$

The second inequality is valid by Lemma 1, then

$$\frac{P(X_1+X_2>x)}{P(X_1>x)+P(X_2>x)}\ge 1+o(1).$$

This completes the proof. ■

Theorem 2. Let X_1 and X_2 be WND random variables with common distribution function $F\in L$. If F satisfies in condition (1), then

$$P(c_1X_1+c_2X_2>x)\sim P(c_1X_1>x)+P(c_2X_2>x)$$
$$as\quad x\to\infty, \quad (6)$$

where $0<a\le c_1,c_2\le b<\infty$.

Proof. Let $c_1=1$, then Lemma 1 implies that X_1,c_2X_2 and c_2X_2 are WND. Suppose that F' be distribution function of c_2X_2 and $0<c_2\le 1$, we get

$$\sup_x \frac{\bar{F}'(x)}{\bar{F}(x)} = \sup_x \frac{P(c_2X_2>x)}{P(X_1>x)} \le 1,$$

therefore, Theorem 1 implies that

$$P(X_1+c_2X_2>x) \sim P(X_1>x)+P(c_2X_2>x) \quad as \quad x\to\infty. \qquad (7)$$

Now, suppose that, $c_2\ge 1$, by (7) we obtain

$$P(X_1+c_2X_2>x)=P(c_3X_1+X_2>x')$$
$$\sim P(c_3X_1>x')+P(X_2>x') \quad (as\ x\to\infty)$$
$$=P(X_1>x)+P(c_2X_2>x),$$

where $c_3=1/c_2\le 1$. Therefore, for any $c\in[a,b]$ we have

$$P(X_1+c\,X_2>x)\sim P(X_1>x)+P(c\,X_2>x) \quad as \quad x\to\infty. \qquad (8)$$

Moreover, for every $c_1,c_2\in[a,b]$, by (6) we get

$$P(c_1X_1+c_2X_2>x)=P(X_1+c'X_2>x')$$
$$\sim P(X_1>x')+P(c'X_2>x') \quad (as\ x\to\infty)$$
$$=P(c_1X_1>x)+P(c_2X_2>x).$$

Where $c'=c_1/c_2$. This completes the proof. ■

Theorem 3. Let $X_1,X_2,\dots,X_n$ be WND random variables with common distribution function $F\in C$. If F satisfies in condition (1), then for any positive real numbers a and b such that $0<a\le b<\infty$,

$$P(\sum_{i=1}^{n} c_iX_i>x)\sim \sum_{i=1}^{n} P(c_iX_i>x) \quad as\ x\to\infty. \qquad (9)$$

where $a\le c_i\le b; i=1,\dots,n$.

Proof. We prove (9) by induction approach. For $n=2$, Theorem 2 implies (9). Now suppose that for any $m\ge 2$

$$P(\sum_{i=1}^{m} c_iX_i>x)\sim \sum_{i=1}^{m} P(c_iX_i>x) \qquad (10)$$

holds when x tends to infinity. Geluk *et al.* [9], show that, when (10) is true for every $x>0$, then

$$P(\sum_{i=1}^{m+1} c_iX_i>x)\le \sum_{i=1}^{m+1} P(c_iX_i>x) \quad as\ x\to\infty. \qquad (11)$$

Since the argument is similar, we omitted it. Moreover, applying Lemma 1, we get

$$P(\sum_{i=1}^{m+1} c_iX_i>x)\ge \sum_{i=1}^{m+1} P(c_iX_i>x) - \sum\sum_{i\ne j} P(c_iX_i>x; c_jX_j>x)$$
$$\ge \sum_{i=1}^{m+1} P(c_iX_i>x) - \sum\sum_{i\ne j} C.P(c_iX_i>x)P(c_jX_j>x) \qquad (12)$$
$$\sim \sum_{i=1}^{m+1} P(c_iX_i>x) \quad as\ x\to\infty.$$

Now (11) and (12) complete the proof. ■

Theorem 4. Let $X_1,X_2,\dots,X_n$ be WND random variables, then

$$P(\max_{1\le i\le n} X_i>x)\sim \sum_{i=1}^{n} P(X_i>x) \quad as\ x\to\infty.$$

Proof. Using Lemma 1, for every $x>0$, we have

$$P(\max_{1\le i\le n} X_i>x)=P(\bigcup_{i=1}^{n}(X_i>x))$$
$$\ge \sum_{i=1}^{n} P(X_i>x) - \sum\sum_{i\ne j} P(X_i>x; X_j>x)$$
$$\ge \sum_{i=1}^{n} P(X_i>x) - \sum\sum_{i\ne j} C.P(X_i>x)P(X_j>x).$$

Applying Lemma 2 we get

$$\sum\sum_{i\ne j} C\,P(X_i>x)P(X_j>x) = o(\sum_{i=1}^{n} P(X_i>x)) \quad as\ x\to\infty.$$

Therefore

$$P(\max_{1\le i\le n} X_i>x)\ge \sum_{i=1}^{n} P(X_i>x).$$

Moreover, it is easy to see that for all $x>0$,

$$P(\max_{1\le i\le n} X_i>x)\le \sum_{i=1}^{n} P(X_i>x).$$

This completes the proof. ■

Theorem 5. Let $X_1, X_2, \dots, X_n$ be WND random variables with common distribution function $F \in C$, which satisfies in the condition (1). Moreover, if $\theta_1, \theta_2, \dots, \theta_n$ are random variables independent of $X_1, X_2, \dots, X_n$ and $P(a \le \theta_i \le b) = 1 ; i = 1, \dots, n$ for some *a* and *b* such that $0 < a \le b < \infty$, then

$$P(\max_{1\le i\le n} \theta_i X_i > x) \sim \sum_{i=1}^{n} P(\theta_i X_i > x) \sim P(\sum_{i=1}^{n} \theta_i X_i > x), \tag{13}$$

when *x* tends to infinity.

Proof. Applying Theorem 4 we get

$$\begin{aligned} P(\sum_{i=1}^{n}\theta_i X_i > x) &= E_{\underline{\theta}}\left[P(\sum_{i=1}^{n}\theta_i X_i > x \,|\underline{\theta})\right] \\ &\sim E_{\underline{\theta}}\left[\sum_{i=1}^{n}P(\theta_i X_i > x\,|\underline{\theta})\right] \quad (as\ x\to\infty) \\ &= \sum_{i=1}^{n} E_{\underline{\theta}}\left[P(\theta_i X_i > x\,|\underline{\theta})\right] \\ &= \sum_{i=1}^{n} P(\theta_i X_i > x). \end{aligned}$$

This gives the second relation in (13). For the first one, applying conditional expectation role and Lemma 1, for every $x > 0$ we have

$$\begin{aligned} P(\max_{1\le i\le n}\theta_i X_i > x) &\ge \sum_{i=1}^{n} P(\theta_i X_i > x) - \sum\sum_{i\ne j} P(\theta_i X_i > x ; \theta_j X_j > x) \end{aligned} \tag{14}$$

$$\begin{aligned} &= \sum_{i=1}^{n} P(\theta_i X_i > x) - \sum\sum_{i\ne j} E_{\theta_i,\theta_j}\left[P(\theta_i X_i > x ; \theta_j X_j > x\,|\theta_i,\theta_j)\right] \\ &\ge \sum_{i=1}^{n} P(\theta_i X_i > x) - \sum\sum_{i\ne j} C . E_{\theta_i}\left[P(\theta_i X_i > x\,|\theta_i)\right] E_{\theta_j}\left[P(\theta_j X_j > x\,|\theta_j)\right] \\ &= \sum_{i=1}^{n} P(\theta_i X_i > x) - \sum\sum_{i\ne j} C . P(\theta_i X_i > x) P(\theta_j X_j > x). \end{aligned}$$

The inequality is valid by independence of X_i and θ_i. Now applying Lemma 2, we get

$$\sum\sum_{i\ne j} C\, P(\theta_i X_i > x) P(\theta_j X_j > x) = o(\sum_{i=1}^{n} P(\theta_i X_i > x)) \quad as\ x\to\infty.$$

Substituting this in (14), we obtain

$$P(\max_{1\le i\le n}\theta_i X_i > x) \ge \sum_{i=1}^{n} P(\theta_i X_i > x) \quad as \quad x\to\infty.$$

This completes the proof. ■

Theorem 6. Let $\{X_j ; j \ge 1\}$ be a sequence of WND random variables with common distribution function *F*, which satisfies in condition (1) and let $\{\theta_j ; j \ge 1\}$ be a sequence of identical positive random variables which is independent of $\{X_j ; j \ge 1\}$. If N is a nonnegative integer-valued random variable independent of sequence $\{X_j ; j \ge 1\}$ and $\{\theta_j ; j \ge 1\}$ with $E(N) < \infty$, then,

$$P(\sum_{i=1}^{N}\theta_i X_i > x) \sim EN . P(\theta_1 X_1 > x) \quad as \quad x\to\infty.$$

Proof. Using Theorem 5, for every $x > 0$ we have

$$\begin{aligned} P(\sum_{i=1}^{N}\theta_i X_i > x) &= E_N\left[P(\sum_{i=1}^{N}\theta_i X_i > x\,|N)\right] \\ &\sim E_N\left[\sum_{i=1}^{N}P(\theta_i X_i > x\,|N)\right] \quad (as\ x\to\infty) \\ &= E_N\left[\sum_{i=1}^{N}P(\theta_i X_i > x)\right] \\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{k}P(\theta_i X_i > x).P(N = k) \\ &= \sum_{i=1}^{\infty}P(\theta_i X_i > x)\sum_{k=i-1}^{\infty}P(N = k) \\ &= \sum_{i=1}^{\infty}P(\theta_i X_i > x).P(N \ge i-1) \\ &= P(\theta_1 X_1 > x).EN . \end{aligned}$$

This completes the proof. ■

Corollary 1. Under the assumptions of Theorem 6, if $P(\theta_i = 1) = 1$ for all $i \ge 1$, then

$$P(\sum_{i=1}^{N} X_i > x) \sim EN . P(X_1 > x) \quad as \quad x\to\infty.$$

Discussion

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with common distribution function *F*, then condition $F(x_1,\ldots,x_n) \leq C.\prod_{i=1}^{n} F_i(x_i)$ can be replaced by $F(x_1,\ldots,x_n) = \prod_{i=1}^{n} F_i(x_i)$. Thus, all above Theorems, Lemmas and Corollaries are true in this case, particularly; Theorem 1 in [8] and Theorem 3.1 in [15] are special case of Theorem 1 and Theorems 3, 4, 5, respectively.

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References

1. Bingham, N.H., Goldie, C.M., Teugels, J.L. *Regular Variation*. Cambridge University Press, Cambridge, UK. (1987).
2. Cai, J. and Tang, Q. On max-sum equivalent and convolution closure of heavy-tailed distributions and their applications. *J.Appl.Prob.*, **41**, 117-130 (2004).
3. Chen, Y. and Kiaw.NG. Tang, Q. Weighted sums of Subexponential random variables and their maxima. *J.Appl.Prob.***37***,* 510-522 (2005).
4. Chen, Y. and Zhang, W. Large deviations for random sums of negatively dependent random variables with consistence varying tails. *Statistics & Probability Letters.***77**, 530-538 (2007).
5. Chistyakov, V.P. A Theorem on sums of independent, positive random variables and its applications to branching processes. *Theory Probability App.* **9**, 640-648 (1964).
6. Cline, D.B.H. and Samorodnitsky, G. Subexponentiality of the product of independent random variables. *Stoch. Process. Their Appl***.** **49**(1): 75-98 (1994).
7. Ebrahimi, N. and Ghosh, M. Multivariate negative dependence. *Commun. Stat. Theory. Math.,* **10,** 307-337 (1981).
8. Embrechts, P. and M.Goldie, C. On closure and factorization properties of Subexponential and related distributions. *J.Austral.Math.Soc.* **29**, 243-256 (1979).
9. Geluk, J.L. and DeVries, C.G. Weighted sums of Subexponential random variables and asymptotic dependence between returns on reinsurance equities. *Mathematics and Economics.* **38,** 39-56 (2006).
10. Jelenkovic, P. and Lazar, A. Asymptotic results for multiplexing subexponential on-off processes. *Adv. Appl. Prob*. **31,** 394-421 (1999).
11. Kiaw. N.G. and Tang, Q. Asymptotic behavior of tail and local probabilities for sums of Subexponential random variables. *J.Appl.Prob*.**41**, 108-116 (2004).
12. Lio, Y. Precise large deviations for negatively associated random variables with consistency varying tails. *Statistics & probability letters*. **77**, 181-189 (2007).
13. Mari, D.D. and Kotz, S. *Correlation and dependence*. Imperial College Press. (2001).
14. Tang, Q. Insensitivity to negative dependence of the asymptotic behavior of precise large deviations. *Electronic Journal of Probability.* **11**, 107-120 (2006).
15. Tang, Q. and Tisitsiashvili, G. Randomly weighted sums of subexponential random variables with application to ruin theory. *Extremes 6*, 171-188 (2003).
16. Wang, D. and Tang, Q. Maxima of sums and random sums for negatively associated random variables with heavy tails. *Statistics & probability letters*. **68**, 287-295 (2004).