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Complete Convergence and Some Maximal Inequalities for Weighted Sums of Random Variables

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Abstract

Let $\{X_n\}$ be a sequence of arbitrary random variables with $EX_n = 0$ and $EX_n^2 < \infty$, for every $n \geq 1$ and $\{a_{nk}\}$ be an array of real numbers. We will obtain two maximal inequalities for partial sums and weighted sums of random variables and also, we will prove complete convergence for weighted sums $\sum_{j=1}^{n} a_{nj}X_j$, under some conditions on a_{nj} and sequence $\{X_n, n \geq 1\}$.

Keywords: Complete convergence; Weighted sums; Maximal inequalities; Pair-wise negative dependence

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robins [4], as follows. A sequence $\{X_n, n \geq 1\}$ of random variables converges completely to a constant a (denoted $\lim_{n\to\infty} X_n = a$, completely), if

$$\sum_{n=1}^{\infty} P[|X_n - a| > \varepsilon] < \infty \quad \text{for all } \varepsilon > 0.$$

From then on, there are many authors who devote the study to the complete convergence for partial sums and weighted sums of independent random variables such as Taylor [13], Hu *et al.* [6], Sung *et al.* [11], Weideng and Zhengran [15] and Sung [12]. Several authors extended this convergence to partial sums and weighted sums of negatively dependent and negatively associated random variables namely Liang and Su [10], Liang [9], Huang and Xu [5] and Amini, and Bozorgnia [1]. In this paper first, we prove two maximal inequalities for partial sums and weighted sums of arbitrary random variables and then present various conditions on $\{a_{nj}\}$ and $\{X_n\}$ for which $\sum_{j=1}^{n} a_{nj}X_j$ converges completely. In addition we consider $\{X_n, n \geq 1\}$ as a sequence of random variables with zero means such that

$$P[|X_n| \geq x] \leq M\int_x^{+\infty} e^{-\gamma t^2}\, dt, \tag{1}$$

for all n and all $x \geq 0$, where M and γ are positive constants. Hanson and Wright [3], obtained a bound on tail probabilities for quadratic forms in independent random variables using condition (1). Wright [16] proved that the bound established by Hanson and Wright [3] for independent symmetric random variables also holds when the random variables are not symmetric but condition (1) is valid.

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Definition. ([2]) The sequence $\{X_n, n \geq 1\}$ of random variables is said to be pair-wise negative dependent (PND) if for every $x_i, x_j \in R$

$$P[X_i \leq x_i, X_j \leq x_j] \leq P[X_i \leq x_i]P[X_j \leq x_j].$$

Lemma 1. ([2]) If the sequence $\{X_n, n \geq 1\}$ is PND, then

$$E(X_i.X_j) \leq E(X_i).E(X_j) \quad \text{for all} \quad i \neq j.$$

Lemma 2. If $Z \sim N(0,1)$, and X satisfies (1), then there exists λ such that

$$EX^2 \leq \lambda^2 EZ^2.$$

Proof. By condition (1), we get

$$EX^2 = \int_0^{+\infty} 2xP[|X| > x]dx$$

$$\leq M\int_0^{+\infty} 2x\left(\int_x^{+\infty} e^{-\gamma t^2} dt\right)$$

$$= M\int_0^{+\infty} t^2 e^{-\gamma t^2} dt \leq \lambda^2 EZ^2,$$

Where $\frac{\sqrt{M}\sqrt[4]{\pi}}{2\gamma^{\frac{3}{4}}} \leq \lambda < \infty$.

Lemma 3. ([8]) The sequence $\{X_n, n \geq 1\}$ converges almost surely if and only if

$$\lim_{n \to \infty} P[\sup_{k \geq 1} |X_{n+k} - X_n| > \varepsilon] = 0,$$

for every $\varepsilon > 0$.

Theorem 1. ([7]) If $\{(X_n, F_n), n \geq 1\}$ is a non-negative sub-martingale, then

$$E(\max_{0 \leq k \leq n} X_k)^p \leq \left(\frac{p}{p-1}\right)^p EX_n^p, \ when \ \ p > 1.$$

2.The Maximal Inequalities

In this section, we prove two maximal inequalities and extend Kolomogorov's convergence criterion of strong law of large numbers and we obtain some other useful results.

Theorem 2. Let $\{X_n, n \geq 1\}$ be a sequence of random variables with $E(X_n) = 0$, $EX_n^2 < \infty$, $n \geq 1$, then for the given $\varepsilon > 0$,

$$P[\max_{1 \leq k \leq n} |S_k| \geq \varepsilon] \leq \frac{32}{\varepsilon^2}\left(\sum_{j=1}^n \sigma_j\right)^2 \tag{2}$$

Where $S_k = \sum_{i=1}^k X_i$ and $\sigma_i = \sqrt{\operatorname{var}(X_i)}$.

Proof. Set $S_{1n} = \sum_{k=1}^n X_k^+$ and $S_{2n} = \sum_{k=1}^n X_k^-$, where $X^+ = \max\{0, X\}$ and $X^- = \max\{0, -X\}$. Since $E[S_{1n}|F_{n-1}] \geq S_{1(n-1)}$, a.e. and $E[S_{2n}|F_{n-1}] \geq S_{2(n-1)}$, a.e. hence the sequences $\{S_{1n}, F_n, n \geq 1\}$ and $\{S_{2n}, F_n, n \geq 1\}$ are nonnegative sub-martingales where $F_n = \sigma(X_1, \cdots, X_n)$ for all $n \geq 1$, where $\sigma(X_1, X_2, \ldots, X_n)$ is the smallest sigma filed produced by $X_1, X_2, \ldots, X_n$. Then, we get by Markov's inequality and Theorem 1 for $p = 2$, that

$$P[\max_{1 \leq k \leq n} S_{1k} > \varepsilon] \leq \frac{1}{\varepsilon^2} E[\max_{1 \leq k \leq n} S_{1k}]^2$$

$$\leq \frac{4ES_{1n}^2}{\varepsilon^2} \leq \frac{4}{\varepsilon^2}\left(\sum_{j=1}^n \sigma_j\right)^2 \quad \text{for all} \quad \varepsilon > 0,$$

the last inequality is true by the following statement

$$ES_{1n}^2 \leq \sum_{k=1}^n \sigma_k^2 + \sum\sum_{i \neq j} \sigma_i \sigma_j = \left(\sum_{k=1}^n \sigma_k\right)^2.$$

Similarly, one can show that

$$P[\max_{1 \leq k \leq n} S_{2k} > \varepsilon] \leq \frac{4}{\varepsilon^2}\left(\sum_{j=1}^n \sigma_j\right)^2 .$$

Combining these two inequalities and $|S_n| \leq S_{1n} + S_{2n}$, we obtain

$$P[\max_{1 \leq k \leq n} |S_k| \geq \varepsilon] \leq P[\max_{1 \leq k \leq n} S_{1k} \geq \frac{\varepsilon}{2}]$$

$$+ P[\max_{1 \leq k \leq n} S_{2k} \geq \frac{\varepsilon}{2}]$$

$$\leq \frac{32}{\varepsilon^2}\left(\sum_{j=1}^n \sigma_j\right)^2 \quad \text{for all} \quad \varepsilon > 0,$$

The following corollary is an extension of Kolomogorov's convergence criterion of strong law of large numbers for arbitrary random variables.

Corollary 1. Let $\{X_n, n \geq 1\}$ be as in Theorem 2.

i) If $\sum_{n=1}^{\infty} \sigma_n < \infty$, then the series $\sum_{n=1}^{\infty} X_n$

converges a.e.

ii) If $\sum_{n=1}^{\infty}\frac{\sigma_n}{b_n}<\infty$, then the following statements hold.

$$\frac{1}{b_n}\sum_{k=1}^{n}X_k \to 0 \quad a.e. \quad as \quad n\to\infty, \tag{3}$$

and

$$E\left(\sup_{n\geq 1}\frac{|S_n|}{b_n}\right)^{\beta}<\infty \quad for\ all\ 0<\beta\leq 2 \tag{4}$$

where $\{b_n\}$ is a sequence of positive increasing real numbers such that

$b_n \to\infty$ as $n\to\infty$.

Proof.

i) By applying Lemma 2, Theorem 2 and Lemma 3, we have

$$P[\sup_{k\geq 1}|S_{n+k} - S_n| > \varepsilon] =$$

$$\lim_{m\to\infty} P[\sup_{1\leq k\leq m}|S_{n+k} - S_n| > \varepsilon]$$

$$\leq \frac{32}{\varepsilon^2}(\sum_{j=n+1}^{\infty}\sigma_j)^2, \quad for\ all \quad \varepsilon>0,$$

Since $\sum_{n=1}^{\infty}\sigma_n<\infty$, it follows that

$$\lim_{m\to\infty} P[\sup_{k\geq 1}|S_{n+k} - S_n| > \varepsilon]=0,$$

this completes the proof.

ii) Taking $Y_n = \frac{X_n}{b_n}$, we get (3) and (4) by Keronecker's Lemma, Lemma 3 and Theorem 2.

Corollary 2. Let $\{X_n, n\geq 1\}$ be as in Theorem 2.

i) If $\sum_{n=1}^{\infty}\sigma_n<\infty$, then for the given $\varepsilon>0$ and for $\alpha>0$, the following statements hold

$$\sum_{n=1}^{\infty}n^{2\alpha-2}P[\max_{1\leq k\leq n}|S_k|>n^{\alpha}\varepsilon]<\infty, \tag{5}$$

and

$$\sum_{n=1}^{\infty}n^{2\alpha-2}P[\sup_{k\geq n}\frac{|S_k|}{k^{\alpha}}>\varepsilon]<\infty, \tag{6}$$

ii) If $\sum_{n=1}^{\infty}n^{\beta-2}\left(\sum_{j=1}^{n}\sigma_j\right)^2<\infty$, for some $\beta>0$, then for every $\varepsilon>0$,

$$\sum_{n=1}^{\infty}n^{\beta}P[\max_{1\leq k\leq n}|S_k|>n\varepsilon]<\infty \quad \text{for all } \varepsilon>0. \tag{7}$$

In particular, if $\sum_{j=1}^{n}\sigma_j = O(n^{1-\frac{\alpha+\beta}{2}})$, for some $\beta>0$ and $\alpha>1$, then we can obtain (5).

iii) If $\{X_n\}$ satisfies condition (1), then $\sum_{j=1}^{n}\alpha_j = O(n)$, and for $0<\beta<1$, we have

$$\sum_{n=1}^{\infty}n^{\beta-2}(\sum_{j=1}^{n}\sigma_j)^2<\infty$$

Theorem 3. Let $\{X_n, n\geq 1\}$ be an arbitrary sequence of random variables with $E(X_n)=0, EX_n^2<\infty, n\geq 1$. Suppose that $\{a_{nj}, 1\leq j\leq n, n\geq 1\}$ be an array of real numbers, then

$$P[\max_{1\leq k\leq n}|T_k|>\varepsilon]\leq \frac{32}{\varepsilon^2}(\sum_{j=1}^{n}|a_{nj}|\sigma_j)^2 \quad \text{for all} \quad \varepsilon>0, \tag{8}$$

where $T_n = \sum_{k=1}^{n}a_{nk}X_k$.

Proof. Set $T_{1n} = \sum_{k=1}^{n}|a_{nk}|X_k^{+}$ and $T_{2n} = \sum_{k=1}^{n}|a_{nk}|X_k^{-}$. Since $E[T_{1n}|F_{n-1}]\geq T_{1(n-1)}$, a.e. and $E[T_{2n}|F_{n-1}]\geq T_{2(n-1)}$, a.e. it follows that the sequences $\{T_{1n}, F_n, n\geq 1\}$ and $\{T_{2n}, F_n, n\geq 1\}$ are nonnegative sub-martingales, where $F_n = \sigma(X_1,\cdots,X_n)$ for all $n\geq 1$. Since $|T_n|\leq T_{1n}+T_{2n}$ for all $n\geq 1$, the proof of (8) follows from the same argument as that in the proof of Theorem 2. Hence

$$P[\max_{1\leq k\leq n}|T_k|\geq\varepsilon]\leq P[\max_{1\leq k\leq n}T_{1k}\geq\frac{\varepsilon}{2}]$$

$$+P[\max_{1\leq k\leq n}T_{2k}\geq\frac{\varepsilon}{2}]$$

$$\leq\frac{32}{\varepsilon^2}(\sum_{j=1}^{n}|a_{nj}|\sigma_j)^2 \quad \text{for all } \varepsilon>0.$$

Corollary 3. Let $\{X_n, n\geq 1\}$ and $\{a_{nj}\}$ be as in Theorem 3,

i) If $\sum_{n=1}^{\infty}n^{\beta-2}(\sum_{j=1}^{n}|a_{nj}|\sigma_j)^2<\infty$, for some $\beta>0$, then we have

$$\sum_{n=1}^{\infty}n^{\beta-2}P[\max_{1\leq k\leq n}|T_k|>\varepsilon]<\infty \quad for\ all \quad \varepsilon>0. \tag{9}$$

ii) If $\{X_n\}$ satisfies condition (1) and $\sum_{j=1}^{n}|a_{nj}| = O(n^{\alpha/2})$ for all $\alpha > 0$, then (9) holds.

3.Complete Convergence for Weighted Sums

Using results of Section 2, we obtain complete convergence for weighted sums $\sum_{j=1}^{n} a_{nj}X_j$ of random variables that satisfy (1) under some conditions on a_{nj}.

Theorem 4. Let $\{X_n, n \geq 1\}$ be an arbitrary sequence of random variables and $\{a_{nj}, 1 \leq j \leq n, n \geq 1\}$ be an array of real numbers such that

$$\sum_{j=1}^{n} |a_{nj} - a_{n(j+1)}| = O(n^{\beta}) \quad \text{for some } \beta > 0,$$

where $a_{n(n+1)} = 0$.

i) If $\{X_n, n \geq 1\}$ satisfies in condition (1), then

$$\sum_{j=1}^{n} a_{nj}X_j \to 0, \qquad \text{completely} \quad \text{as } n \to \infty. \tag{10}$$

ii) If $\sum_{k=1}^{n} \sigma_k = O(n^{\alpha})$ for some $\alpha > 0$, then (10) holds.

Proof. Using Abel's partial summation rule we get

$$|\sum_{j=1}^{n} a_{nj}X_j| \leq \max_{1\leq i\leq n}|S_i|\,(\sum_{j=1}^{n}|a_{nj} - a_{n(j+1)}|) \leq Cn^{-\beta}\max_{1\leq i\leq n}|S_i| \quad \text{for some } \beta > 0. \tag{11}$$

From Theorem 2, we conclude that

$$\sum_{n=1}^{\infty} P(n^{-\beta}\max_{1\leq i\leq n}|S_i| > \varepsilon) \leq \frac{32}{\varepsilon^2}\sum_{n=1}^{\infty} n^{-2\beta}\left(\sum_{j-1}^{n}\sigma_i\right)^2 \quad \text{for all } \varepsilon > 0. \tag{12}$$

Hence (11) and (12) yield

$$\sum_{n=1}^{\infty} P[|\sum_{j=1}^{n} a_{nj}X_j| > \varepsilon] \leq \sum_{n=1}^{\infty} P(n^{-\beta}\max_{1\leq i\leq n}|S_i| > \varepsilon) \leq \frac{32}{\varepsilon^2}\sum_{n=1}^{\infty} n^{-2\beta}\left(\sum_{j=1}^{n}\sigma_j\right)^2.$$

i) Condition (1) implies that

$$\sum_{n=1}^{\infty} n^{-2\beta}\left(\sum_{j=1}^{n}\sigma_j\right)^2 \leq \lambda^2\sum_{n=1}^{\infty}\frac{1}{n^{2\beta-2}} < \infty \quad \text{for all } \beta > 3/2.$$

ii) If $\sum_{k=1}^{n}\sigma_k = O(n^{\alpha})$ for some $\alpha > 0$, then

$$\sum_{n=1}^{\infty} n^{-2\beta}\left(\sum_{j=1}^{n}\sigma_j\right)^2 < M\sum_{n=1}^{\infty}\frac{1}{n^{2\beta-2\alpha}} < \infty \quad \text{for all } \beta > 1/2 + \alpha.$$

Where $0 < M < \infty$, this completes the proof.

Theorem 5. Let $\{X_n, n \geq 1\}$ be an arbitrary sequence of random variables with zero means that satisfies (1) and $\{a_{nj}\}$ be as in Theorem 4. Then, there exists $\lambda > 0$ such that

$$P[|\sum_{j=1}^{n} a_{nj}X_j| > \varepsilon] \leq \frac{\lambda^2}{\varepsilon^2}\left(\sum_{j=1}^{n}|a_{nj}|\right)^2 \quad \text{for all } \varepsilon > 0.$$

Proof. Applying Markov's inequality, Cauchy-Shwarz's inequality and Lemma 2, we have

$$P[|\sum_{j=1}^{n} a_{nj}X_j| > \varepsilon] \leq \frac{1}{\varepsilon^2}E\left(\sum_{j=1}^{n} a_{nj}X_j\right)^2$$

$$\leq \frac{1}{\varepsilon^2}\left(\sum_{j=1}^{n}(|a_{nj}|)^2 EX_j^2 + \sum\sum_{i\neq j}|a_{ni}|a_{nj}|E|X_iX_j|\right)$$

$$\leq \frac{1}{\varepsilon^2}\left(\sum_{j=1}^{n}(|a_{nj}|)^2 EX_j^2 + \sum\sum_{i\neq j}|a_{ni}||a_{nj}|\sqrt{EX_i^2 EX_j^2}\right)$$

$$\leq \frac{1}{\varepsilon^2}\left(\sum_{j=1}^{n}|a_{nj}|\sigma_j\right)^2$$

$$\leq \frac{\lambda^2}{\varepsilon^2}\left(\sum_{j=1}^{n}|a_{nj}|\right)^2 \qquad \text{for all } \varepsilon > 0.$$

Corollary 4. Let $\{X_n, n \geq 1\}$ and $\{a_{nj}\}$ be as in Theorem 5.

i) If $\sum_{i=1}^{n}|a_{ni}| = O(1)$, then for all $\beta > \frac{1}{2}$,

$$n^{-\beta}\sum_{j=1}^{n} a_{nj}X_j \to 0, \qquad \text{completely} \quad \text{as } n \to \infty.$$

ii) If $\max_{1\le i\le n} |a_{ni}| = O(n^{\frac{-\delta}{2}})$, $0 < \delta < 2$, then for all $\beta > 1 - \frac{\delta}{2}$, we have

$$n^{-\beta}\sum_{j=1}^{n} a_{nj}X_j \to 0, \quad \text{completely as } n \to \infty.$$

Theorem 6. Let $\{X_n, n \ge 1\}$ be a sequence of PND random variables with zero means that satisfies (1). Let $\{a_{nj}\}$ be an array of positive real numbers with $\sum_{i=1}^{n} a_{ni}^2 = O(n^{\delta})$, for all $\delta \ge 0$. Then for all $\beta > \frac{1+\delta}{2}$, we have

$$n^{-\beta}\sum_{j=1}^{n} a_{nj}X_j \to 0, \quad \text{completely as } n \to \infty.$$

Proof. By Markov's inequality, Lemmas 1 and 2, we get

$$\sum_{n=1}^{\infty} P[n^{-\beta}|\sum_{j=1}^{n} a_{nj}X_j| > \varepsilon]$$

$$\le \sum_{n=1}^{\infty} \frac{1}{n^{2\beta}\varepsilon^2} E\left(\sum_{j=1}^{n} a_{nj}X_j\right)^2$$

$$\le \sum_{n=1}^{\infty} \frac{1}{n^{2\beta}\varepsilon^2}\left(\sum_{j=1}^{n} a_{nj}^2 EX_j^2 + \sum\sum_{i\ne j} a_{nj}a_{nj}EX_iEX_j\right)$$

$$\le \sum_{n=1}^{\infty} \frac{\lambda^2}{n^{2\beta}\varepsilon^2}\sum_{j=1}^{n} a_{nj}^2$$

$$\le \sum_{n=1}^{\infty} \frac{\lambda^2}{n^{2\beta-\delta}\varepsilon^2} < \infty \quad \textit{for all } \varepsilon > 0.$$

This completes the proof.

4. Examples

In the following we have several examples that satisfy the conditions of Theorem 2, and Corollaries 1 and 2.

1. Let $\{X_n, n \ge 1\}$ be a sequence of arbitrary random variables.

i) If $X_n \sim \exp(\lambda_n)$ for all $n \ge 1$ and $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, then $\sum_{n=1}^{\infty} X_n$ converges a.e. In particular if $\lambda_n = n^{\alpha}$ for all $\alpha > 1$, then $\sum_{n=1}^{\infty} X_n$ converges a.e.

ii) Let $P[X_n = 0] = 1 - \frac{1}{n^{\alpha}}$ and $P[X_n = \mp n] = \frac{1}{2n^{\alpha}}$ for all $\alpha > 4$. Since $\sum_{n=1}^{\infty} \sigma_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{\alpha}{2}-1}} < \infty$, it follows that $\sum_{n=1}^{\infty} X_n$ converges a.e.

iii) If $X_n \sim U(-a_n, a_n)$, $0 < a_n < 1$ for all $n \ge 1$, and $\sum_{n=1}^{\infty} a_n < \infty$, then $\sum_{n=1}^{\infty} X_n$ converges a.e.

iv) Let $X_n \sim \Gamma(m_0, n^{\alpha})$, for all $\alpha > 0$. Since $\sum_{n=1}^{\infty} \frac{\sigma_n}{n} < \infty$, it follows that

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - EX_k) \to 0 \quad \text{a.e.}$$

2. Let $\{X_n\}$ be a sequence of i.i.d. random variables with distribution $U[0,1]$. Set $Y_n = \prod_{k=1}^{n} X_k$ and $\sigma_n = \sqrt{\mathrm{Var}(Y_n)}$. It is obvious that $EY_n = \frac{1}{2^n}$ and $\mathrm{Var}(Y_n) < \frac{1}{3^n}$, for all $n \ge 1$. Since $\sum_{n=1}^{\infty} \sigma_n < \infty$, from Corollary 1.i, we conclude that $\sum_{n=1}^{\infty}(Y_n - EY_n)$ converges a.e. Next, note that $\sum_{n=1}^{\infty} EY_n < \infty$ a.e. hence

$$\sum_{n=1}^{\infty}\prod_{k=1}^{n} X_k, \quad \text{converges a.e.}$$

Also, the conditions of Corollary 2.i are valid for the sequence $\{Y_n\}$. Thus (5) and (6) hold.

3. Let $\{X_n\}$ be a sequence of random variables with the probability function,

$$P[X_n = 0] = P[X_n = 2] = \frac{1}{2} \quad \text{for all } n \ge 1.$$

It is obvious that conditions of Corollaries 1 and 2 are valid for the sequence $\{\frac{X_n}{3^n}\}$, hence $\sum_{n=1}^{\infty} \frac{X_n}{3^n}$ converges a.e. and the statements (5) and (6) are true.

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