Journal of Sciences, Islamic Republic of Iran 18(4): 329-337 (2007) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Nonlinear Two-Phase Stefan Problem

K. Ivaz[1,*] and N. Aliyev[2]

[1]Department of Mathematics, University of Tabriz, Tabriz, Islamic Republic of Iran
[2]Department of Applied Mathematics and Economical Cybernetics,
Baku State University, 370148 Baku, Azerbaijan

Abstract

In this paper we consider a nonlinear two-phase Stefan problem in one-dimensional space. The problem is mapped into a nonlinear Volterra integral equation for the free boundary.

Keywords: Free boundary problems; Integral equation

Introduction

Processes related to solidification (or melting) are important in many engineering applications such as freezing of foodstuff, casting of alloys, preservation of human blood, solar energy storage and many others. The study on the moving boundary problems involving solidification (or melting) has become a highly popular subject recently. These problems are inherently nonlinear due to the presence of moving boundary solution.

One and two phase Stefan problems for the linear heat equation have been the subject of many studies in the past [1,2]. Indeed these problems have a great physical relevance since they provide a mathematical model for the processes of phase changes [3,4].

The boundary between the two phases is a free boundary its motion has to be determined as part of the solution.

In [5,6] exact solution were found in parametric form for a class of Stefan problems in nonlinear heat conduction. More recently the previous analysis [7-13] was extended to nonlinear diffusion models.

It is the aim of this paper to analyse a two-phase Stefan problem for the nonlinear heat equation. Such an equation arises as a model of heat conduction in solidification [14,15].

In the next section we show that the two-phase Stefan problem for the nonlinear heat equation reduces to a nonlinear Volterra integral equation.

Statement of the Problem

Let

$$\Omega_l = \left\{(x,t)\,\middle|\,0 < x < \gamma(t),\ t > t\right\},$$

$$\Omega_s = \{(x,t)\,|\,\gamma(t) < x < 1, t > 0\}$$

be the liquid and solid domains, respectively. We start our analysis with the following system of nonlinear heat equations:

$$(2.0.1)\ \frac{\partial u_l}{\partial t} - \frac{\partial}{\partial x}\left[a(u_l)\frac{\partial u_l}{\partial x}\right] = 0, (x,t) \in \Omega_l,$$

$$(2.0.2)\ \frac{\partial u_s}{\partial t} - \frac{\partial}{\partial x}\left[a(u_s)\frac{\partial u_s}{\partial x}\right] = 0, (x,t) \in \Omega_s,$$

$$(2.0.3)\ u_l(x,0) = u_{l_0}(x), 0 \le x \le b = \gamma(0),$$

$$(2.0.4)\ u_s(x,0) = u_{s_0}(x), b \le x \le 1,$$

* Corresponding author, Tel.: +98(411)3342102, Fax: +98(411)3342102, E-mail: ivaz@tabrizu.ac.ir

(2.0.5) $u_l(0,t)=f_1(t), t\ge 0,$

(2.0.6) $u_s(1,t)=f_2(t), t\ge 0,$

(2.0.7) $u_l(\gamma(t),t)=u_s(\gamma(t),t)=0, t\ge 0,$

(2.0.8) $$\frac{\partial u_s(x,t)}{\partial x}\Big|_{x=\gamma(t)}-\frac{\partial u_l(x,t)}{\partial x}\Big|_{x=\gamma(t)}=-\gamma'(t), t\ge 0,$$

where $a(u_l), a(u_s), u_{l_0}, u_{s_0}, f_1$ and f_2 are assumed to be known and $u_l(x,t)$, $u_s(x,t)$ and the free boundary $\gamma(t)$ are unknown and must be determined. We also suppose that

(2.0.9) $a(0)=0.$

If we set

(2.0.10) $$\begin{aligned} a(u_l(x,t)) &= a(u_{l_0}(x)) \\ &+[a(u_l(x,t))-a(u_{l_0}(x))] \\ &=\alpha_l(x)+A_l(x,u_l(x,t)), \end{aligned}$$

(2.0.11) $$\begin{aligned} a(u_s(x,t)) &= a(u_{s_0}(x)) \\ &+[a(u_s(x,t))-a(u_{s_0}(x))] \\ &=\alpha_s(x)+A_s(x,u_s(x,t)), \end{aligned}$$

it is clear that, we can write

(2.0.12) $$\alpha_l(x)=\alpha_s(x)=\alpha(x), \quad (x,t)\in\Gamma\equiv\{x=\gamma(t)|0<t\},$$

(2.0.13) $A_l(\gamma(t),0)=A_s(\gamma(t),0)=-\alpha(\gamma(t)).$

We also suppose that

(2.0.14) $\alpha_l(0)=a(f_1(0)), \alpha_s(1)=a(f_2(0)),$

then we can write (2,0,1)-(2,0,2) as follow:

(2.0.15) $$\begin{aligned} lu_l &\equiv \frac{\partial u_l}{\partial t}-\frac{\partial}{\partial x}[\alpha_l(x)\frac{\partial u_l}{\partial x}] \\ &=\frac{\partial}{\partial x}[A_l(x,u_l)\frac{\partial u_l}{\partial x}], (x,t)\in\Omega_l, \end{aligned}$$

(2.0.16) $$\begin{aligned} lu_s &\equiv \frac{\partial u_s}{\partial t}-\frac{\partial}{\partial x}[\alpha_s(x)\frac{\partial u_s}{\partial x}] \\ &=\frac{\partial}{\partial x}[A_s(x,u_s)\frac{\partial u_s}{\partial x}], (x,t)\in\Omega_s, \end{aligned}$$

We also define

$$(lu,v)\equiv(lu_l,v_l)+(lu_s,v_s)$$

$$\equiv\int_{\Omega_l}\frac{\partial u_l(x,t)}{\partial t}v_l(x,t)d\Omega_l$$

$$+\int_{\Omega_s}\frac{\partial u_s(x,t)}{\partial t}v_s(x,t)d\Omega_s$$

$$-\int_{\Omega_l}\frac{\partial}{\partial x}[\alpha_l(x)\frac{\partial u_l}{\partial x}]v_l(x,t)d\Omega_l$$

$$-\int_{\Omega_s}\frac{\partial}{\partial x}[\alpha_s(x)\frac{\partial u_s}{\partial x}]v_s(x,t)d\Omega_s$$

$$=\int_{\Omega_l}\frac{\partial}{\partial x}[A_l(x,u_l)\frac{\partial u_l(x,t)}{\partial x}]v_l(x,t)d\Omega_l$$

$$+\int_{\Omega_s}\frac{\partial}{\partial x}[A_s(x,u_s)\frac{\partial u_s(x,t)}{\partial x}]v_s(x,t)d\Omega_s$$

Now, integrating by parts, we obtain

$$\int_{\partial\Omega_l}u_l(x,t)v_l(x,t)\cos(\nu,t)d\partial\Omega_l$$

$$-\int_{\Omega_s}u_l(x,t)\frac{\partial v_s(x,t)}{\partial t}d\Omega_l$$

$$+\int_{\partial\Omega_{ls}}u_s(x,t)v_s(x,t)\cos(\nu,t)d\partial\Omega_s$$

$$-\int_{\Omega_s}u_s(x,t)\frac{\partial v_s(x,t)}{\partial t}d\Omega_s$$

$$-\int_{\partial\Omega_l}[\alpha_l(x)\frac{\partial u_l(x,t)}{\partial x}v_l(x,t)$$

$$-u_l(x,t)\alpha_l(x)\frac{\partial v_l(x,t)}{\partial x}]\cos(\nu,x)d\partial\Omega_l$$

$$-\int_{\Omega_l}u_l(x,t)\frac{\partial}{\partial x}[\alpha_l(x)\frac{\partial v_l(x,t)}{\partial x}]d\Omega_l$$

$$-\int_{\partial\Omega_s}[\alpha_s(x)\frac{\partial u_s(x,t)}{\partial x}v_s(x,t)$$

$$-u_s(x,t)\alpha_s(x)\frac{\partial v_s(x,t)}{\partial x}]\cos(\nu,x)\,d\,\partial\Omega_s$$

$$-\int_{\Omega_s} u_s(x,t)\frac{\partial}{\partial x}[\alpha_s(x)\frac{\partial v_s(x,t)}{\partial x}]d\,\Omega_s$$

$$=\int_{\partial\Omega_l} A_l(x,u_l)\frac{\partial u_l(x,t)}{\partial x}v_l(x,t)\cos(\nu,x)\,d\,\partial\Omega_l$$

$$-\int_{\Omega_l} A_l(x,u_l)\frac{\partial u_l(x,t)}{\partial x}\,\frac{\partial v_l(x,t)}{\partial x}d\,\Omega_l$$

$$+\int_{\partial\Omega_s} A_s(x,u_s)\frac{\partial u_s(x,t)}{\partial x}v_s(x,t)\cos(\nu,x)\,d\,\partial\Omega_s$$

$$-\int_{\Omega_s} A_s(x,u_s)\frac{\partial u_s(x,t)}{\partial x}\,\frac{\partial v_s(x,t)}{\partial x}d\,\Omega_s\,.$$

By choosing

$v_s(\gamma(t),t)=v_l(\gamma(t),t)=v(\gamma(t),t),$

and by (2.0.3)-(2.0.8) we obtain

$$-\int_0^b u_{l_0}(x)v_l(x,0)dx$$

$$+\int_0^\infty u_l(\gamma(t),t)v_l(\gamma(t),t)\gamma'(t)dt$$

$$+\lim_{t\to\infty}\int_0^{\gamma(\infty)} u_l(x,t)v_l(x,t)dx$$

$$+\int_0^\infty u_s(\gamma(t),t)v_s(\gamma(t),t)\gamma'(t)dt$$

$$-\int_b^1 u_{s_0}(x)v_s(x,0)dx$$

$$+\lim_{t\to\infty}\int_{\gamma(\infty)}^1 u_s(x,t)v_s(x,t)dx$$

$$+\int_0^\infty[\alpha_l(0)\frac{\partial u_l(x,t)}{\partial x}\Big|_{x=0}v_l(0.t)$$

$$-u_l(0,t)\alpha_l(0)\frac{\partial v_l(x,t)}{\partial x}\Big|_{x=0}]dt$$

$$-\int_0^\infty[\alpha_l(\gamma(t))\frac{\partial u_l(x,t)}{\partial x}\Big|_{x=\gamma(t)}v_l(\gamma(t),t)$$

$$-u_l(\gamma(t),t)\alpha_l(\gamma(t))\frac{\partial v_l(x,t)}{\partial x}\Big|_{x=\gamma(t)}]dt$$

$$+\int_0^\infty[\alpha_s(\gamma(t))\frac{\partial u_s(x,t)}{\partial x}\Big|_{x=\gamma(t)}v_s(\gamma(t),t)$$

$$-u_s(\gamma(t),t)\alpha_s(\gamma(t))\frac{\partial v_s(x,t)}{\partial x}\Big|_{x=\gamma(t)}]dt$$

$$-\int_0^\infty[\alpha_s(1)\frac{\partial u_s(x,t)}{\partial x}\Big|_{x=1}v_s(1,t)$$

$$-u_s(1,t)\alpha_s(1)\frac{\partial v_s(x,t)}{\partial x}\Big|_{x=1}]dt$$

$$+\int_0^\infty A_l(0,u_l(0,t))\frac{\partial u_l(x,t)}{\partial x}\Big|_{x=0}v_l(0,t)dt$$

$$-\int_0^\infty[A_s(\gamma(t),u_l(\gamma(t),t)$$

$$\frac{\partial u_s(x,t)}{\partial x}\Big|_{x=\gamma(t)}v_l(\gamma(t),t)dt$$

$$+\int_0^\infty[A_s(\gamma(t),u_s(\gamma(t),t))$$

$$\frac{\partial u_s(x,t)}{\partial x}\Big|_{x=\gamma(t)}v_s(\gamma(t),t)dt$$

$$-\int_0^\infty[A_s(1,u_s(1,t))\frac{\partial u_s(x,t)}{\partial x}\Big|_{x=1}v_s(1,t)dt$$

$$+\int_{\Omega_l} A_l(x,u_l)\frac{\partial u_l(x,t)}{\partial x}\frac{\partial v_l(x,t)}{\partial x}d\,\Omega_l$$

$$+\int_{\Omega_s} A_s(x,u_s)\frac{\partial u_s(x,t)}{\partial x}\frac{\partial v(x,t)}{\partial x}d\,\Omega_s$$

$$=\int_{\Omega_l} u_l(x,t)[\frac{\partial v(x,t)}{\partial t}$$

$$+\frac{\partial}{\partial x}(\alpha_l(x)\frac{\partial v_l(x,t)}{\partial x})]d\,\Omega_l$$

$$+\int_{\Omega_s} u_s(x,t)[\frac{\partial v_s(x,t)}{\partial t}$$

$$+\frac{\partial}{\partial x}(\alpha_s(x)\frac{\partial v_s(x,t)}{\partial x})]d\Omega_s$$

(2.0.17)

We note that

$$\lim_{t\to\infty}\int_0^{\gamma(\infty)} u_l(x,t)v_l(x,t)dx$$

$$+\lim_{t\to\infty}\int_{\gamma(\infty)}^1 u_s(x,t)v_s(x,t)dx = 0$$

$$-\int_0^\infty \alpha(\gamma(t))\frac{\partial u_l(x,t)}{\partial x}\Big|_{x=\gamma(t)} v_l(\gamma(t),t)dt$$

$$+\int_0^\infty \alpha_s(\gamma(t))\frac{\partial u_s(x,t)}{\partial x}\Big|_{x=\gamma(t)} v_s(\gamma(t),t)dt$$

$$=-\int_0^\infty \alpha(\gamma(t))v(\gamma(t),t)\gamma'(t)dt$$

We now suppose that $V_l(x,\xi;t-\tau)$ and $V_s(x,\xi;t-\tau)$ are the fundamental solutions of the following problem in x direction, respectively

$$\frac{\partial V_l(x,\xi;t-\tau)}{\partial t}$$

$$+\frac{\partial}{\partial x}[\alpha_l(x)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}] \tag{2.0.18}$$

$$=\delta(x-\xi)\delta(t-\tau),(x,t)\in\Omega_l$$

$$\frac{\partial V_s(x,\xi;t-\tau)}{\partial t}$$

$$+\frac{\partial}{\partial x}[\alpha_s(x)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}] \tag{2.0.19}$$

$$=\delta(x-\xi)\delta(t-\tau),(x,t)\in\Omega_s$$

By using (2,0,16) and (2,0,1)-(2,0,6) we find that first important relation for the solution of two-phase Stefan problem for the nonlinear heat equation is given by:

$$-\int_0^b u_{l_0}(x)V_l(x,\xi;-\tau)dx$$

$$-\int_b^1 u_{s_0}(x)V_s(x,\xi;-\tau)dx$$

$$+\int_0^\infty[\alpha_l(0)\frac{\partial u_l(x,t)}{\partial x}\Big|_{x=0}V_l(0,\xi;t-\tau)$$

$$-u_l(0,t)\alpha_l(0)\frac{\partial V_l(x,\xi;t-\tau t)}{\partial x}\Big|_{x=0}]dt$$

$$-\int_0^\infty \alpha(\gamma(t))V_l(\gamma(t),\xi;t-\tau)\gamma'(t)dt$$

$$-\int_0^\infty[\alpha_s(1)\frac{\partial u_s(x,t)}{\partial x}\Big|_{x=1}V_s(1,\xi;t-\tau)$$

$$-u_s(1,t)\alpha_l(1)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}\Big|_{x=1}]dt$$

$$+\int_0^\infty A_l(0,f_1(t))\frac{\partial u_l(x,t)}{\partial x}\Big|_{x=0}V_l(0,\xi;t-\tau)dt$$

$$-\int_0^\infty A_l(\gamma(t),0)\frac{\partial u_l(x,t)}{\partial x}\Big|_{x=\gamma(t)}V_l(\gamma(t),\xi;t-\tau)dt$$

$$+\int_0^\infty[A_s(\gamma(t),0)\frac{\partial u_s(x,t)}{\partial x}\Big|_{x=\gamma(t)}V_s(\gamma(t),\xi;t-\tau)dt$$

$$-\int_0^\infty[A_s(1,f_2(t))\frac{\partial u_s(x,t)}{\partial x}\Big|_{x=1}V_s(1,\xi;t-\tau t)dt$$

$$+\int_{\Omega_{1l}} A_l(x,u_l(x,t))\frac{\partial u_l(x,t)}{\partial x}\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}d\Omega_l$$

$$+\int_{\Omega s} A_s(x,u_s(x,t))\frac{\partial u_s(x,t)}{\partial x}\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}d\Omega_s$$

$$=\begin{cases} u_l(\xi,\tau) & (\xi,\tau)\in\Omega_l, \\ \frac{1}{2}u_l(\xi,\tau) & (\xi,\tau)\in\partial\Omega_l, \\ u_s(\xi,\tau) & (\xi,\tau)\in\Omega_s, \\ \frac{1}{2}u_s(\xi,\tau) & (\xi,\tau)\in\partial\Omega_s. \end{cases} \tag{2.0.20}$$

Therefore, by (2,0,20) we can compute the u_l and u_s at the domains and boundaries of domains. It is clear that the values of u_l and u_s at the domains and boundaries of domains are dependent to the following parameters

$$\frac{\partial u_l(x,t)}{\partial x}\Big|_{x=0}, \quad \frac{\partial u_s(x,t)}{\partial x}\Big|_{x=1},$$

$$u_l(x,t), u_s(x,t), \frac{\partial u_l(x,t)}{\partial x}, \frac{\partial u_s(x,t)}{\partial x},$$

$$for\ (x,t)\in\Omega_l\ and\ (x,t)\in\Omega_s .$$

We now try to obtain the following

$$\frac{\partial u_l(x,t)}{\partial x}\Big|_{x=0}, \frac{\partial u_s(x,t)}{\partial x}\Big|_{x=1},$$

For this purpose, we do similar work to [16]-[19] as follows:

$$(lu,\frac{\partial V}{\partial x}) \equiv (lu_l,\frac{\partial V_l}{\partial x}) + (lu_s,\frac{\partial V_s}{\partial x})$$

$$\equiv \int_{\Omega_l} \frac{\partial u_l(x,t)}{\partial t}\frac{\partial V_l(x,\xi;t-\tau)}{\partial x} d\Omega_l$$

$$+ \int_{\Omega_s} \frac{\partial u_s(x,t)}{\partial t}\frac{\partial V_s(x,\xi;t-\tau)}{\partial x} d\Omega_s$$

$$- \int_{\Omega_l} \frac{\partial}{\partial x}[\alpha_l(x)\frac{\partial u_l(x,t)}{\partial x}]\frac{\partial V_l(x,\xi;t-\tau)}{\partial x} d\Omega_l$$

$$- \int_{\Omega s} \frac{\partial}{\partial x}[\alpha_s(x)\frac{\partial u_s(x,t)}{\partial x}]\frac{\partial V_s(x,\xi;t-\tau)}{\partial x} d\Omega_s$$

$$= \int_{\Omega_l} \frac{\partial}{\partial x}[A_l(x,u_l)\frac{\partial u_l(x,t)}{\partial x}]$$

$$\frac{\partial V_l(x,\xi;t-\tau)}{\partial x} d\Omega_l$$

$$+ \int_{\Omega_s} \frac{\partial}{\partial x}[A_s(x,u_s)\frac{\partial u_s(x,t)}{\partial x}]$$

$$\frac{\partial V_s(x,\xi;t-\tau)}{\partial x} d\Omega_s .$$

We now try the derivatives of functions $u_l(x,t), u_s(x,t), V_l(x,t;t-\tau)$ and $V_s(x,t;t-\tau)$ in the domains that do not appear higher than 1 on the boundaries $\partial\Omega_l$ and $\partial\Omega_s$ and do not appear higher than 2 in the domains.

For this purpose, we do as follows:

$$\int_{\partial\Omega_l} u_l(x,t)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}\cos(\nu,t)\, d\,\partial\Omega_l$$

$$- \int_{\Omega_s} u_l(x,t)\frac{\partial^2 V_l(x,\xi;t-\tau)}{\partial x\,\partial t} d\Omega_l$$

$$+ \int_{\partial\Omega_s} u_s(x,t)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}\cos(\nu,t)\, d\,\partial\Omega_s$$

$$- \int_{\Omega s} u_s(x,t)\frac{\partial^2 V_s(x,\xi;t-\tau)}{\partial x\,\partial t} d\Omega_s$$

$$- \int_{\partial\Omega_l} [\alpha_l(x)\frac{\partial u_l(x,t)}{\partial x}]$$

$$\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}\cos(\nu,x)\, d\,\partial\Omega_l$$

$$+ \int_{\Omega_{1l}} \alpha_l(x)\frac{\partial u_l(x,t)}{\partial x}\frac{\partial^2 V_l(x,\xi;t-\tau)}{\partial x^2} d\Omega_l$$

$$- \int_{\partial\Omega_s} [\alpha_s(x)\frac{\partial u_s(x,t)}{\partial x}]$$

$$\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}\cos(\nu,x)\, d\,\partial\Omega_s$$

$$+ \int_{\Omega_s} \alpha_s(x)\frac{\partial u_s(x,t)}{\partial x}\frac{\partial^2 V_s(x,\xi;t-\tau)}{\partial x^2} d\Omega_s$$

$$= \int_{\partial\Omega_l} [A_l(x,u_l(x,t))\frac{\partial u_l(x,t)}{\partial t}]$$

$$\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}\cos(\nu,x)\, d\,\partial\Omega_l$$

$$- \int_{\Omega_s} [A_l(x,u_l(x,t))\frac{\partial u_l(x,t)}{\partial x}]$$

$$\frac{\partial^2 V_l(x,\xi;t-\tau)}{\partial x^2}d\Omega_l$$

$$+\int_{\partial\Omega_s}[A_s(x,u_s(x,t))\frac{\partial u_s(x,t)}{\partial t}]$$

$$\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}\cos(\nu,x)d\partial\Omega_s$$

$$-\int_{\Omega_s}[A_s(x,u_s(x,t))\frac{\partial u_s(x,t)}{\partial x}]$$

$$\frac{\partial^2 V_s(x,\xi;t-\tau)}{\partial x^2}d\Omega_s$$

And integration by parts, we obtain

$$\int_{\partial\Omega_l}u_l(x,t)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}\cos(\nu,t)d\partial\Omega_l$$

$$-\int_{\partial\Omega_l}u_l(x,t)\frac{\partial V_l(x,\xi;t-\tau)}{\partial t}\cos(\nu,x)d\partial\Omega_l$$

$$+\int_{\Omega_1}\frac{\partial u_l(x,t)}{\partial x}\frac{\partial V_l(x,\xi;t-\tau)}{\partial t}d\Omega_l$$

$$+\int_{\partial\Omega_s}u_s(x,t)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}\cos(\nu,t)d\partial\Omega_s$$

$$-\int_{\partial\Omega s}u_s(x,t)\frac{\partial V_s(x,\xi;t-\tau)}{\partial t}\cos(\nu,x)d\partial\Omega_s$$

$$+\int_{\Omega_s}\frac{\partial u_s(x,t)}{\partial x}\frac{\partial V_s(x,\xi;t-\tau)}{\partial t}d\Omega_s$$

$$-\int_{\partial\Omega_l}\alpha_l(x)\frac{\partial u_l(x,t)}{\partial x}$$

$$\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}\cos(\nu,x)d\partial\Omega_l$$

$$+\int_{\Omega_1}\frac{\partial u_l(x,t)}{\partial x}[\alpha_l(x)\frac{\partial^2 V_l(x,\xi;t-\tau)}{\partial x^2}$$

$$+\alpha_l'(x)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}$$

$$-\alpha_l'(x)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]d\Omega_l$$

$$-\int_{\partial\Omega s}\alpha_s(x)\frac{\partial u_s(x,t)}{\partial x}\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}$$

$$\cos(\nu,x)d\partial\Omega_s$$

$$+\int_{\Omega_s}\frac{\partial u_s(x,t)}{\partial x}[\alpha_s(s)\frac{\partial^2 V_s(x,\xi;t-\tau)}{\partial x^2}$$

$$+\alpha_s'(x)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}$$

$$-\alpha_s'(x)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]d\Omega_s$$

$$=\int_{\partial\Omega_l}A_l(x,u_l(x,t))\frac{\partial u_l(x,t)}{\partial t}\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}$$

$$\cos(\nu,x)d\partial\Omega_l$$

$$-\int_{\Omega_s}\frac{\partial u_l(x,t)}{\partial x}\frac{A_l(x,u_l(x,t))}{\alpha_l(x)}[\delta(x-\xi)\delta(t-\tau)$$

$$-\frac{\partial V_l(x,\xi;t-\tau)}{\partial t}-\alpha_l'(x)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]d\Omega_l$$

$$+\int_{\partial\Omega s}A_s(x,u_s(x,t))\frac{\partial u_s(x,t)}{\partial t}\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}$$

$$\cos(\nu,x)d\partial\Omega_s$$

$$-\int_{\Omega_s}\frac{\partial u_s(x,t)}{\partial x}\frac{A_s(x,u_s(x,t))}{\alpha_s(x)}[\delta(x-\xi)\delta(t-\tau)$$

$$-\frac{\partial V_s(x,\xi;t-\tau)}{\partial t}-\alpha_s'(x)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]d\Omega_s$$

By using of (2.0.3)-(2.0.7), (2.0.17)-(2.0.18) and some computations, we can write

$$\lim_{t\to\infty}\int_0^{\gamma(\infty)}u_l(x,t)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}dx$$

$$+\lim_{t\to\infty}\int_{\gamma(\infty)}^1 u_s(x,t)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}dx=0,$$

$$\int_0^\infty[u_l(x,t)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]\Big|_{x=\gamma(t)}\gamma'(t)dt=0,$$

$$\int_0^\infty [u_l(x,t)\frac{\partial V_l(x,\xi;t-\tau)}{\partial t}]\Big|_{x=\gamma(t)} dt = 0,$$

$$\int_0^\infty [u_s(x,t)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]\Big|_{x=\gamma(t)} \gamma'(t)dt = 0,$$

$$-\int_0^\infty [u_s(x,t)\frac{\partial V_s(x,\xi;t-\tau)}{\partial t}]\Big|_{x=\gamma(t)} dt = 0,$$

$$\int_0^\infty \alpha_l(\gamma(t))[\frac{\partial u_l(x,t)}{\partial x}\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]\Big|_{x=\gamma(t)} dt$$

$$-\int_0^\infty \alpha_s(\gamma(t))[\frac{\partial u_s(x,t)}{\partial x}\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]\Big|_{x=\gamma(t)} dt$$

$$=\int_0^\infty \alpha(\gamma(t))\frac{\partial V(x,\xi;t-\tau)}{\partial x}\Big|_{x=\gamma(t)} \gamma'(t)dt,$$

and

$$\int_0^\infty [A_l(x,u_l(x,t))$$

$$\frac{\partial u_l(x,t)}{\partial t}\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]\Big|_{x=\gamma(t)} dt$$

$$-\int_0^\infty [A_s(x,u_s(x,t))$$

$$\frac{\partial u_s(x,t)}{\partial x}\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]\Big|_{x=\gamma(t)} dt$$

$$=-\int_0^\infty \alpha(\gamma(t))\frac{\partial V(x,\xi;t-\tau)}{\partial x}\Big|_{x=\gamma(t)} \gamma'(t)dt$$

A applying above relation, we now obtain a similar relation to (2.0.20) for computing the values of unknown function at the boundary of domain. For this purpose we do as follows:

$$\int_0^b [u_{l_0}(x)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]\Big|_{t=0} dx$$

$$-\int_0^\infty [f_1(t)\frac{\partial V_l(x,\xi;t-\tau)}{\partial t}]\Big|_{x=0} dt$$

$$+\int_0^b [u_{s_0}(x)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]\Big|_{t=0} dx$$

$$-\int_0^\infty [f_2(t)\frac{\partial V_s(x,\xi;t-\tau)}{\partial t}]\Big|_{x=1} dt$$

$$-\int_0^\infty [\alpha_l(0)\frac{\partial u_l(x,t)}{\partial x}\Big|_{x=0}\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]\Big|_{x=0} dt$$

$$+\int_0^\infty [\alpha_s(1)\frac{\partial u_s(x,t)}{\partial x}\Big|_{x=1}\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]\Big|_{x=1} dt$$

$$-\int_0^\infty [A_l(x,u_l(x,t))$$

$$\frac{\partial u_l(x,t)}{\partial x}\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]\Big|_{x=0} dt$$

$$+\int_0^\infty [A_s(x,u_s(x,t))$$

$$\frac{\partial u_s(x,t)}{\partial x}\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]\Big|_{x=1} dt$$

$$+\int_{\Omega_l}\frac{\partial u_l(x,t)}{\partial x}\alpha_l'(x)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}d\Omega_l$$

$$+\int_{\Omega_s}\frac{\partial u_s(x,t)}{\partial x}\alpha_s'(x)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}d\Omega_s$$

$$+\int_{\Omega_l}\frac{\partial u_l(x,t)}{\partial x}[\frac{\partial V_l(x,\xi;t-\tau)}{\partial t}$$

$$+\alpha_l'(x)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]\frac{A_l(x,u_l(x,t))}{\alpha_l(x)}d\Omega_l$$

$$+\int_{\Omega_s}\frac{\partial u_s(x,t)}{\partial x}[\frac{\partial V_s(x,\xi;t-\tau)}{\partial t}$$

$$+\alpha_s'(x)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]\frac{A_s(x,u_s(x,t))}{\alpha_s(x)}d\Omega_s$$

$$(2.0.21) = \begin{cases} \dfrac{a(u_l(\xi,\tau))}{\alpha_l(\xi)}\dfrac{\partial u_l(\xi,\tau)}{\partial \xi} & (\xi,\tau)\in\Omega_l, \\ \dfrac{1}{2}\dfrac{a(u_l(\xi,\tau))}{\alpha_l(\xi)}\dfrac{\partial u_l(\xi,\tau)}{\partial \xi} & (\xi,\tau)\in\partial\Omega_l, \\ \dfrac{a(u_s(\xi,\tau))}{\alpha_s(\xi)}\dfrac{\partial u_s(\xi,\tau)}{\partial \xi} & (\xi,\tau)\in\Omega_s, \\ \dfrac{1}{2}\dfrac{a(u_s(\xi,\tau))}{\alpha_s(\xi)}\dfrac{\partial u_s(\xi,\tau)}{\partial \xi} & (\xi,\tau)\in\partial\Omega_s. \end{cases}$$

We thus proved:

Theorem. Let the function $a(u)$ in problem (2.0.1)-(2.0.8) that satisfies (2.0.9), (2.0.14) and $V_l(x,\xi;t-\tau)$, $V_s(x,\xi;t-\tau)$ be the fundamental solution of (2.0.17) and (2.0.18), respectively. Then the solution of problem (2.0.1)-(2.0.8) satisfies (2.0.20) and (2.0.21).

By using (2.0.20), we can obtain

$$\frac{1}{2}[\frac{a(u_l(\xi,\tau))}{\alpha_l(\xi)}\frac{\partial u_l(\xi,\tau)}{\partial \xi}]\Big|_{\xi=0}$$

as follows:

$$\frac{1}{2}[\frac{a(u_l(\xi,\tau))}{\alpha_l(\xi)}\frac{\partial u_l(\xi,\tau)}{\partial \xi}]\Big|_{\xi=0}$$

$$=\int_0^b [u_{l_0}(x)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]\Big|_{t=0,\xi=0}dx$$

$$-\int_0^\infty [f_1(t)\frac{\partial V_l(x,\xi;t-\tau)}{\partial t}]\Big|_{x=0,\xi=0}dt$$

$$+\int_b^1 [u_{s_0}(x)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]\Big|_{t=0,\xi=0}dx$$

$$-\int_0^\infty [f_2(t)\frac{\partial V_s(x,\xi;t-\tau)}{\partial t}]\Big|_{x=1,\xi=0}dt$$

$$-\int_0^\infty [A_l(x,u_l(x,t))$$

$$\frac{\partial u_l(x,t)}{\partial t}\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]\Big|_{x=0,\xi=0}dt$$

$$+\int_0^\infty [A_s(x,u_s(x,t))$$

$$\frac{\partial u_s(x,t)}{\partial x}\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]\Big|_{x=1,\xi=0}dt$$

$$+\int_{\Omega_l}\frac{\partial u_l(x,t)}{\partial x}\alpha_l'(x)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}\Big|_{\xi=0}d\Omega_l$$

$$+\int_{\Omega s}\frac{\partial u_s(x,t)}{\partial x}\alpha_s'(x)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}\Big|_{\xi=0}d\Omega_s$$

$$+\int_{\Omega_l}\frac{\partial u_l(x,t)}{\partial x}[\frac{\partial V_l(x,\xi;t-\tau)}{\partial t}$$

$$+\alpha_l'(x)\frac{\partial V_l(x,\xi;t-\tau)}{\partial x}]\Big|_{\xi=0}\frac{A_l(x,u_l(x,t))}{\alpha_l(x)}d\Omega_l$$

$$+\int_{\Omega_s}\frac{\partial u_s(x,t)}{\partial x}[\frac{\partial V_s(x,\xi;t-\tau)}{\partial t}$$

$$+\alpha_s'(x)\frac{\partial V_s(x,\xi;t-\tau)}{\partial x}]\Big|_{\xi=0}\frac{A_s(x,u_s(x,t))}{\alpha_s(x)}d\Omega_s$$

(2.0.22)

In similar fusion we obtain $\frac{1}{2}[\frac{a(u_s(\xi,\tau))}{\alpha_s(\xi)}\frac{\partial u_s(\xi,\tau)}{\partial \xi}]\Big|_{\xi=1}$ and call it (2.0.23).

Now, considering (2.0.21) and (2.0.22) we obtain a system of Fredholm integral equation for the $\frac{\partial u_l(\xi,\tau)}{\partial \xi}\Big|_{\xi=0}$ and $\frac{\partial u_s(\xi,\tau)}{\partial \xi}\Big|_{\xi=1}$ and call it (2.0.24) and (2.0.25), respectively.

We also obtain some similar relation for $\frac{\partial u_l(\xi,\tau)}{\partial \xi}\Big|_{\xi=0}$ and $\frac{\partial u_s(\xi,\tau)}{\partial \xi}\Big|_{\xi=1}$ in domains Ω_l and Ω_s.

Thus by (2.0.19)-(2.0.25) we will obtain a system of nonlinear Fredholm integral equation of the second kind for u_l and u_s. Hence, by substitution of u_l and u_s in (2.0.8), we will obtain a nonlinear Fredholm integral equation for the free boundary $\gamma(t)$.

References

1. Rubinstein L.I. The Stefan problem. American Mathematical Society Translations, Vol. 27, American Mathematical Society, Providence, RI, (and refs. therein) (1971).
2. Friedman A. Free boundary problems for parabolic equations. I. Melting of solids. *J. Math. Mech.*, **8**: 499-517 (1959).
3. Kolodner I. Free boundary problem for the heat equation with applications to problems of change of phase. *Comm. Pure Appl. Math.*, **9**: 1-31 (1956).
4. Crank J. *Free and Moving Boundary Problems*. Clarendon, Oxford (1984).
5. Rogers C. Application of Reciprocal Transformation to a Two-phase Stefan Problem. *J. Phys.,* A18, L105-L109 (1985).
6. Natale M.F. and Tarzia D.A. Explicit Solutions to the Two-Phase Stefan Problem for Storm-Type Materials. *J.*

Phys., **A33**: 395-404 (2000).
7. Rogers C. *J. Phys.*, A18, L105 (1985).
8. Rogers C. *J. Nonlinear Mech.*, 21: 249 (1986).
9. Ablowitz M.J. and Delillo S. Nonliniearity. **13**: 471 (2000).
10. Ablowitz M.J. and Delillo S. *Phys. Lett.*, **A271**: 273 (2000).
11. Ablowitz M.J. and Delillo S. *J. Phys. A: Math. Gen.*, **35**: 1 (2003).
12. Delillo S. and Salvatori M.C. *J. Nonlinear Math. Phys.*, **9**: 446 (2002).
13. Delillo S., Salvatori M.C., and Sanchini G. *Phys. Lett.*, **A310**: 25 (2003).
14. Tien R.H. and Geiger G.E. A heat-transfer analysis os the solidification of a binary eutectic systems. *J. Heat Trans. ASME*, **89**: 230-4 (1967).
15. Voller V.R. Development and application of a heat balance integral method for analysis of metallurgical solodification. *Appl. Math. Model*, **13**: 3-11 (1989).
16. Bahrami F., Aliev N., and Hosseini S.M. A Method for the reduction of four dimensional mixed problems with general boundary conditions to a systems of second kind fredholm integral equations. *Italian Journal of Pure and Applied Mathematics-N.*, **17**: 91-104 (2005).
17. Hosseini S.M. and Aliev N. Sufficient conditions for the reduction of a BVP for PDE with non-local and global boundary conditions to Fredholm integral equatins (on a rectangular domain). *Appl. Math. Comput.*, **147**: 664-685 (2004).
18. Aliev N. and Jahanshahi M. Solution of Poisson's equation with global, local and non-local bondary conditions. *Int. J. Math. Educ. Sci. Technol.*, **33**(2): 241-247 (2002).
19. Aliev N. and Hosseini S.M. An analysis of a parabolic problem with general (non-local and global) supplementary linear conditions. *Italian Journal of Pure and Applied Mathematics-N.*, **12**: 143-154 (2002).