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Derivations on Certain Semigroup Algebras

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Abstract

In the present paper we give a partially negative answer to a conjecture of Ghahramani, Runde and Willis. We also discuss the derivation problem for both foundation semigroup algebras and Clifford semigroup algebras. In particular, we prove that if S is a topological Clifford semigroup for which E_S is finite, then $H^1(M(S),M(S))=\{0\}$.

Keywords: Foundation semigroup; Semigroup algebra; Derivation; First order cohomology; Clifford semigroup

1. Introduction

Let S be a locally compact topological semigroup, and let $\mathrm{M}(\mathrm{S})$ denote the space of all bounded complex regular measures on S. This space with the convolution product and norm $\|\mu\| = |\mu|(S)$ is a Banach algebra. The space of all measures $\mu \in M_a(S)$ for which the mappings $s \mapsto \delta_s * |\mu|$ and $s \mapsto |\mu| * \delta_x$ from s into $\mathrm{M}(\mathrm{S})$ are weakly continuous is denoted by $M_a(S)$ (or $\mathrm{L}(\mathrm{S})$ as in [1]), where δ_s denotes the Dirac measure at s. Note that the measure algebra $M_a(S)$ defines a two-sided closed L -ideal of $\mathrm{M}(\mathrm{S})$ (see [1]).

For a locally compact topological semigroup S, let

$M_0(S) := \{\mu \in M(S) : \mu(S) = 0\}$ and

$I_0(S) = M_a(S) \cap M_0(S)$.

A semigroups S is called a *foundation semigroup*; if $\cup\{\mathrm{supp}(\mu) : \mu \in M_a(S)\}$ is dense in S. Note that if S is a foundation semigroup with an identity then $M_a(S)$ has a bounded approximate identity (c.f. [16]).

Let S be a foundation semigroup. Given any $\mu \in M_a(S)$ and $\phi \in M_a(S)^*$, define the complex-valued function $\phi \circ \mu$ and $\mu \circ \phi$ on S by

$(\phi \circ \mu)(s) = \phi(\delta_s * \mu)$ and

$(\mu \circ \phi)(s) = \phi(\mu * \delta_s) \quad (s \in S)$.

It is clear that $\phi \circ \mu$ and $\mu \circ \phi$ are in $\mathrm{C_b}(\mathrm{S})$, where $\mathrm{C_b}(\mathrm{S})$ denotes the space of all bounded continuous complex-valued functions on S. By Lemma 3.4 of [16], for each $\phi \in M_a(S)^*$ and $\mu, \nu \in M_a(S)$, $\phi(\mu * \nu) = \nu(\mu \circ \phi) = \mu(\phi \circ \nu)$.

Let S be a Banach algebra and X be a Banach A -bimodule. A bounded linear map $D : A \to X$ is called an X *-derivation*, if

$D(ab) = D(a).b + a.D(b) \quad (a, b \in A)$.

For every $x \in X$ we define $\mathrm{ad_x}$ by

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$$\mathrm{ad}_x(a) = a.x - x.a \quad (a \in A).$$

It is easily seen that ad_x is a derivation. Derivations of this form are called *inner derivations*. The set of all derivations from A into X is denoted by $Z^1(A,X)$, and the set of all inner X-derivations is denoted by $B^1(A,X)$. Clearly, $Z^1(A,X)$ is a linear subspace of the space of all bounded linear operators of A into X and $B^1(A,X)$ is a linear subspace of $Z^1(A,X)$. We denote by $H^1(A,X)$ the difference space of $Z^1(A,X)$ modulo $B^1(A,X)$.

It is a conjecture raised by Ghahramani, Runde and Willis that a semigroup S for which $H^1(M(S), M(S)) = \{0\}$ must satisfy some form of cancellation property; see [8]. In this paper, we study this problem for a certain class of topological semigroups and give a partially negative answer to this conjecture.

2. Some Examples and Results

Let A be a commutative Banach algebra. The spectrum of A, denoted by $\mathrm{Spec}(A)$, is the set of all multiplicative linear functionals on A. The radical of A is defined by

$\mathrm{Rad}(A) = \cap\{\ker(\phi) : \phi \in \mathrm{Spec}(A)\}$. Recall that a commutative Banach algebra A is called *semisimple* if $\mathrm{Rad}(A) = 0$.

Example 2.1, Following Leinert [14], let S be the semigroup of all sequences $s = (x_i)$ of real numbers x_i such that $x_i > 0$ for almost all i with pointwise addition. Then $\ell^1(S)$ is a commutative semisimple Banach algebra, and so from Theorem 16.21 of [2] it follows that $H^1(\ell^1(S), \ell^1(S)) = \{0\}$.

Remark 2.2. Let S be a locally compact semigroup with the semigroup structure of $st = s_0$ for all $s, t \in S$ where s_0 is a fixed element of S. Clearly for $\mu, \nu \in M(S)$, $\mu * \nu = \mu(S)\nu(S)\delta_{s_0}$. Let $\phi \in \mathrm{Spec}(M(S))$, then $\phi(\mu) = \mu(S)$ for all $\mu \in M(S)$. Indeed, if $\phi \in \mathrm{Spec}(M(S))$, then for all $\mu \in M(S)$

$$\mu(S)\phi(\delta_{s_0}) = \phi(\mu(S)\delta_{s_0})$$
$$= \phi(\mu * \delta_{s_0}) = \phi(\mu)\phi(\delta_{s_0}).$$

Now if $\phi(\delta_{s_0}) = 0$, then $\phi(\mu) = 0$ for all $\mu \in M(S)$. This contradiction shows that $\phi(\delta_{s_0}) \neq 0$, and so $\phi(\mu) = \mu(S)$ all $\mu \in M(S)$. Therefore

$$\mathrm{Rad}(M(S)) = \bigcap_{\phi \in \mathrm{Spec}(M(S))} \ker(\phi) = M_0(S) \neq 0.$$

This implies that $M(S)$ is not semisimple. Also $H^1(M(S), M(S))$ is not zero (c.f. [8], Example on page 387). Thus the hypothesis of semisimplity in Theorem 16.21 of [2] is necessary.

In the following we give examples of a semigroup S for which the first order cohomology $H^1(M(S), M(S)) = \{0\}$, but S is neither left and nor right cancellative. This is a partially negative answer to the guess of Ghahramani, Runde and Willis in [8].

Example 2.3. Let A be a non-empty set, and let $S = A \cup \{0\}$. With the multiplication defined by $s^2 = s$ and $st = 0$ for all $s, t \in S$ with $s \neq t$, S is a commutative semigroup. Since for each $t \in A$, the function ϕ_t defined by $\phi_t(s) = 0$ for $s \neq t$ and $\phi_t(t) = 1$ is a semicharacter on S, so the set of all semicharacters on S, separates the points of S. Hence by Proposition 4.1.4 of [6], $\ell^1(S)$ is semisimple. From Theorem 16.21 of [2], it follows that $H^1(\ell^1(S), \ell^1(S)) = 0$, although S is not either left or right cancellative.

Remark 2.4. Let S be a compact, Hausdorff, cancellative right topological semigroup, then S is a compact topological group and so $H^1(M(S), M(S)) = \{0\}$.

Before proving our next theorem we first need to prove two lemmas.

Lemma 2.5. Let S be a locally compact left zero semigroup with $\mathrm{Card}(S) > 2$. Then S is a right cancellative semigroup for which $H^1(M(S), M(S)) \neq \{0\}$

Proof. Suppose first that S is a locally compact left zero semigroup, then it is clear that S is a right cancellative. Clearly for $\mu, \nu \in M(S)$, $\mu * \nu = \nu(S)\mu$. Moreover we have

$$Z^1(M(S), M(S)) =$$
$$\{L \in B(M(S), M(S)) : L(M(S)) \subseteq M_0(S)\}.$$

To see this, take $D \in Z^1\big(M(S), M(S)\big)$ and $\mu \in M(S)$, then

$$\begin{aligned}\mu(S)D(\mu) &= D\big(\mu(S)\mu\big)\\ &= D(\mu * \mu)\\ &= D(\mu)*\mu + \mu * D(\mu)\\ &= \mu(S)D(\mu) + D(\mu)(S)\,\mu.\end{aligned}$$

Thus $D(\mu)(S) = 0$. This implies that $D(\mu) \in M_0(S)$.

Conversely, if $D \in B\big(M(S), M(S)\big)$, such that $D\big(M(S)\big) \subseteq M_0(S)$, then

$$\begin{aligned}D(\mu * \nu) &= D\big(\nu(S)\mu\big)\\ &= \nu(S)D(\mu)\\ &= \nu(S)D(\mu) + D(\nu)(S)\,\mu\\ &= D(\mu)*\nu + \mu * D(\nu).\end{aligned}$$

Now since $\mathrm{Card}(S) \geq 3$, there exist $s_1, s_2, s_3 \in S$ such that $s_i \neq s_j$ for $i \neq j$. By the Hahn-Banach theorem there exists $D \in B\big(M(S), M_0(S)\big)$ such that $D(\delta_{s_1}) = 0$ and $D(\delta_{s_2}) = \delta_{s_3} - \delta_{s_1}$ (indeed, by the Hahn-Banach theorem there exists $\bar{D} \in B\big(M(S), \mathbb{C}(\delta_{s_3} - \delta_{s_1})\big)$ that extends the following bounded linear map,

$$\mathbb{C}\delta_{s_1} \oplus \mathbb{C}\delta_{s_2} \to \mathbb{C}(\delta_{s_3} - \delta_{s_1}) : \lambda_1\delta_{s_1} + \lambda_2\delta_{s_2} \mapsto \lambda_2(\delta_{s_3} - \delta_{s_1}),$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$. Now, define $D \in B\big(M(S), M_0(S)\big)$ by $D(\mu) = \bar{D}(\mu)\ (\mu \in M(S))$. By (1), D is a derivation. If $D = \mathrm{ad}_\nu$ for some $\nu \in M(S)$, then $D(\mu) = \nu(S)\,\mu - \mu(S)\,\nu$. This implies that

$$\begin{aligned}0 = D(\delta_{s_1}) &= \nu(S)\delta_{s_1} - \delta_{s_1}(S)\nu\\ &= \nu(S)\delta_{s_1} - \nu,\end{aligned}$$

and so $\nu = \nu(S)\delta_{s_1}$. Similarly $\delta_{s_3} - \delta_{s_1} = D(\delta_{s_2}) = \nu(S)\delta_{s_2} - \nu$. Therefore $\delta_{s_3} - \delta_{s_1} = \nu(S)(\delta_{s_2} - \delta_{s_1})$, and hence

$$\begin{aligned}1 &= (\delta_{s_3} - \delta_{s_1})(\{s_3\})\\ &= (\nu(S)(\delta_{s_2} - \delta_{s_1}))(\{s_3\}) = 0.\end{aligned}$$

This contradiction shows that $D \notin B^1\big(M(S), M(S)\big)$. Thus $H^1\big(M(S), M(S)\big) \neq \{0\}$. □

Lemma 2.6. Let S be a left zero semigroup with $\mathrm{Card}(S) = 2$, then $H^1\big(M(S), M(S)\big) = \{0\}$.

Proof. Let $S = \{s, t\}$ and $D \in Z^1\big(M(S), M(S)\big)$. Then from (1) it follows that $D(M(S)) \subseteq M_0(S)$. Suppose that $D(\delta_s) = \alpha(\delta_s - \delta_t)$ and $D(\delta_t) = \beta(\delta_s - \delta_t)$. Set $\phi = \alpha\delta_t - \beta\delta_s$, then

$$\mathrm{ad}_\phi(\delta_s) = \delta_s * \phi - \phi * \delta_s = \alpha(\delta_s - \delta_t).$$

Thus $\mathrm{ad}_\phi(\delta_s) = D(\delta_s)$. Similarly $\mathrm{ad}_\phi(\delta_t) = D(\delta_t)$. So $ad_\phi = D$. □

A combination of the above two lemmas yields the following result.

Theorem 2.7. Let S be a left zero semigroup. Then $H^1\big(M(S), M(S)\big) = \{0\}$ if and only if $\mathrm{Card}S \leq 2$.

Remark 2.8. Let S be a left zero semigroup with two elements. Then by Lemma 2.6 we have $H^1\big(M(S), M(S)\big) = \{0\}$, but by Proposition 2.5 we have $H^1\big(M(S \times S), M(S \times S)\big) \neq \{0\}$.

3. Derivations on Foundation Semigroups

Our starting point of this section is the following definition.

Definitions 3.1. If a Banach algebra A is contained in another Banach algebra B as a closed ideal, then the *strict topology* or *strong operator topology* (*so*) on B with respect to A is defined through the family of seminorms $(p_a)_{a \in A}$, where

$$p_a(b) := \|ba\| + \|ab\| \quad (b \in B).$$

For a topological semigroup S the strict topology on $M(S)$ with respect to $\mathrm{M}_a(\mathrm{S})$ is simply called *the so topology* or *the strict topology* on $\mathrm{M(S)}$.

Lemma 3.2. Let B be a Banach algebra and A be an ideal of B. Then $T \in (B, so)^*$ if and only if there exits subset $\{a_1, a_2, ..., a_n\}$ and $\{a_1^{'}, ..., a_m^{'}\}$ of A and $\{\phi_1, ..., \phi_n\}$ and $\{\phi_1^{'}, ..., \phi_m^{'}\}$ of A^* such that for each $b \in B$

$$T(b) = \sum_{i=1}^{m} \phi_i (a_i b) + \sum_{i=1}^{n} \phi_i^{'} (ba_i^{'}).$$

Proof. Let $T \in (B, so)^*$. Then by Theorem 3.1 of [4] there exist a_1, $a_2, ..., a_n$ in A, such that

$$|T(b)| \le \sum_{i=1}^{n} (\|a_i b\| + \| ba_i \|) \quad (b \in B).$$

Let $M = \{(a_1 b, ..., a_n b, ba_1, ..., ba_n) : b \in B\}$, and define the functional $F_0 : M \mapsto \mathbb{C}$ by

$$F_0(a_1 b, ..., a_n b, ba_1, ..., ba_n) = T(b).$$

Clearly $M \subseteq \oplus_{i=1}^{2n} A$ A$ and F_0 is well defined and bounded. By the Hahn-Banach theorem there is a bounded functional F on $\oplus_{i=1}^{n} A$ such that $F|_M = F_0$. For all $1 \le i \le n$ and $1 \le j \le 2$, define $\phi_{ij} \in A^*$ by

$$\phi_{ij}(a) = F(0, ..., \overset{(j-1)n+i}{\overset{\frown}{a}}, ..., 0) \qquad (a \in A).$$

Now for any $b \in B$ we have

$$\begin{aligned} T(b) &= F_0(a_1 b, ..., a_n b, ba_1, ..., ba_n) \\ &= F(a_1 b, ..., a_n b, ba_1, ..., ba_n) \\ &= \sum_{i=1}^{n} \phi_{i1}(a_i b) + \sum_{i=1}^{n} \phi_{i2}(ba_i). \end{aligned}$$

The other side is trivial. □

The following result is a generalization of Proposition 3.3.41(i) of [5] from locally compact groups to the case of foundation semigroups with completely different technique of proof.

Theorem 3.3. Let S be a foundation semigroup. Then $\ell^1(S)$ is so-dense in $M(S)$.

Proof. We may isometrically imbed $M(S)$ into $C_b(S)^*$. By Lemma 2.5 of [1], with the weak* topology on $C_b(S)^*$, $\ell^1(S)$ is dense in $C_b(S)^*$. Now suppose $\mu \in M(S)$. Then there exists a net (μ_α) in $\ell^1(S)$, such that $\mu_\alpha \to \mu$ in the weak* topology. Now let $\phi \in M_a(S)^*$ and $\nu \in M_a(S)$, then

$$\phi(\mu_\alpha * \nu) = \mu_\alpha(\nu \circ \phi) \to \mu(\nu \circ \phi) = \phi(\mu * \nu).$$

Therefore by Lemma 3.2 for any $T \in (M(S), so)^*$ we have $T(\mu_\alpha) \to T(\mu)$. So $\ell^1(S)$ is weakly dense in the locally convex space $(M(S), so)$. Since $\ell^1(S)$ is convex, by Theorem 3.12 of [15] we have $\mu \in \overline{\ell^1(S)}^{so}$. □

Proposition 3.4. Let S be a foundation semigroup with identity. Then $D(M_a(S)) \subseteq M_a(S)$ for any $D \in Z^1(M_a(S), M(S))$

Proof. Let (e_α) be a bounded approximate identity for $M_a(S)$, then for each $\mu \in M_a(S)$,

$$\begin{aligned} D(\mu) &= \lim_\alpha D(\mu * e_\alpha) \\ &= \lim_\alpha (D(\mu) * e_\alpha + \mu * D(e_\alpha)) \in M_a(S). \end{aligned}$$

□

Recall that S is said to be *left compactly cancelletive* if $C^{-1}D$ is a compact subset of S for all compact subsets C and D of S, where

$$C^{-1}D = \{x \in S : cx \in D \text{ for some } c \in C\}.$$

Right compactly cancellative locally compact semigroups are defined similarly. A semigroup which is both left and right compactly cancellative is called *compactly cancellative*.

Let A be a Banach algebra. A pair (L, R) of operators L and R on A is called a multiplier if for each $a, b \in A$, $L(ab) = L(a)b, R(ab) = aR(b)$ and $aL(b) = R(a)b$. The set of all multipliers on A, denoted by $M(A)$ with the multiplication defined by

$$\begin{aligned} &(L_1, R_1)(L_2, R_2) \\ &= (L_1 \circ L_2, R_2 \circ R_1) \quad ((L_1, R_1), (L_2, R_2) \in M(A)), \end{aligned}$$

is a Banach algebra that called *the multiplier algebra* of A.

In the proof of the following lemma we have been inspired by that of Theorem 3.3.40 of [5].

Lemma 3.5. Let S be a compactly cancellative foundation semigroup with identity, Then the multiplier algebra of $M_a(S)$ is isomorphic with $M(S)$.

Proof. For $\mu \in M(S)$, define

$L_\mu(\nu) = \mu * \nu$ and

$R_\mu(\nu) = \nu * \mu \quad (\nu \in M_a(S))$.

Clearly (L_μ, R_μ) is a multiplier of $M_a(S)$. We show that the mapping $\mu \mapsto (L_\mu, R_\mu)$ is an isomorphism from $M(S)$ onto the multiplier algebra of $M_a(S)$. Let (e_α) be a bounded approximate identity for $M_a(S)$, and (L, R) be a multiplier of $M_a(S)$, then $(L(e_\alpha))$ is a bounded net in $M(S)$. By Banach-Alaoglu's Theorem, passing to a subnet if necessary, we can assume that there exists $\mu \in M(S)$, such that $L(e_\alpha) \to \mu$ in the weak* topology. Let $\nu \in M_a(S)$ and $\phi \in C_0(S)$. By Lemma 1 of [12], $\phi \circ \nu \in C_0(S)$. So

$$\begin{aligned}\lim_\alpha \langle \phi, L(e_\alpha) * \nu \rangle &= \lim_\alpha \langle \nu \circ \phi, L(e_\alpha) \rangle \\ &= \langle \nu \circ \phi, \mu \rangle \\ &= \langle \phi, \mu * \nu \rangle \\ &= \langle \phi, L_\mu(\nu) \rangle,\end{aligned}$$

and hence $L(e_\alpha) * \nu \to L_\mu(\nu)$ in the weak* topology. Now, since $L(e_\alpha * \nu) \to L(\nu)$ in the norm topology, we have $L = L_\mu$. Similarly $R = R_\mu$. The remainder of proof is trivial. □

Proposition 3.6. Let S be a compactly cancellative foundation semigroup with identity, Then $H^1(M(S), M(S)) = H^1(M_a(S), M(S))$. Furthermore each $D \in Z^1(M_a(S), M(S))$ has a unique so-weak* continuous extension $\bar{D} \in Z^1(M(S), M(S))$.

Proof. From Lemma 3.5 the set of all multipliers on $M_a(S)$ is equal with $M(S)$. On the other hand, by Lemma 1 of [12] we have $M_a(S) \circ C_0(S) \subseteq C_0(S)$. Also, let (e_α) be a bounded approximate identity for $M_a(S)$. As in Lemma 2.1 from [12],

$$\| e_\alpha \circ f - f \|_\infty \to 0 \qquad (f \in C_0(S)).$$

Thus $M_a(S) \circ C_0(S) = C_0(S)$ by Cohen factorization theorem. Similarly, $C_0(S) \circ M_a(S) = C_0(S)$. Therefore $C_0(S)$ is a neo-unital $M_a(S)$-module. By Propositions 1.9 and 1.11 from [10] the proof is complete. □

4. Derivations on Clifford Semigroups

An element e of a semigroup S is called an idempotent if $e^2 = e$. We denote be E_S the set of idempotents in S. Recall that a semigroup S is a *Clifford semigroup* if it is an inverse semigroup for which each idempotent is central (cf. [9], 4.2). By Theorem 4.2.1 of [9], S is a semilattice of groups and if $S = \cup\{G_e : e \in E_S\}$, then for $e, f \in E$, $e \le f$ if and only if $ef = f$, and moreover for every $e, f \in E$, $G_e G_f \subseteq G_{ef}$.

Lemma 4.1. Let S be a topological Clifford semigroup, and $D \in Z^1(M(S), M(S))$, then $D(\ell^1(S)) \subseteq M_0(S)$.

Proof. Suppose that $S = \cup_{e \in E_S} G_e$. Let $x \in S$, then there exists $e \in E$ such that $x \in G_e$. If H is a subgroup of G_e\$G_e\$ generated by x and e, then H is abelian and therefore $\ell^1(H)$ is amenable. We note that $M(S)$ is a $\ell^1(H)$-bimodule and the restriction of D on $\ell^1(H)$ denoted by D_x is a derivation. Thus D_x\$ is inner. That is there is $\mu_x \in M(S)$ such that $D_x = ad_{\mu_x}$. Therefore for any $x \in H$, we have $D_x(\delta_x) = \delta_x * \mu_x - \mu_x * \delta_x$ and so that $D(\delta_x) = \delta_x * \mu_x - \mu_x * \delta_x$. Thus $D(\delta_x)(S) = 0$. This implies that $D(\ell^1(S)) \subseteq M_0(S)$. □

The following theorem is a generalization of Proposition 7.1 of [8].

Theorem 4.2. Let S be a compactly cancellative foundation Clifford semigroup with identity and $D \in Z^1(M_a(S), M(S))$, then $D(M_a(S)) \subseteq I_0(S)$.

Proof. By Proposition 3.6, D has a unique extension $\bar{D} \in Z^1(M(S), M(S))$. Using Theorem 3.3 and Lemmas 3.6 and 4.1 we obtain

$$D\left(M_a(S)\right) \subseteq \bar{D}\left(M(S)\right) = \bar{D}\left(\overline{\ell^1(S)}^{so}\right)$$
$$\subseteq \overline{\bar{D}\left(\ell^1(S)\right)}^{weak^*}$$
$$= \overline{M_0(S)}^{weak^*} = M_0(S).$$

On the other hand by Proposition 3.4 $D\left(M_a(S)\right) \subseteq M_a(S)$, thus $D\left(M_a(S)\right) \subseteq I_0(S)$. □

Remark 4.3. (a) Let T be a compact foundation semilattice with identity, for example $T = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$ with the $k.l = max\{k, l\}$ $(k, l \in T)$. Let G be any locally compact group. Then $S = T \times G$ with the product topology and coordinatewise multiplication defines a foundation semigroup (See [7], Page 43) with identity that is compactly cancelllative. Let $G_t = \{t\} \times G$ for $t \in T$. It is clear that G_t is a group with the identity (t, e_G). Clearly $S = \bigcup_{e \in T} G_e$ and S is a Clifford semigroup. Furthermore $E_S = \{(t, e_G) : t \in T\}$. By Theorem 4.2, if $D \in Z^1\left(M_a(S), M(S)\right)$, then $D\left(M_a(S)\right) \subseteq I_0(S)$.

(b) The proof of the Theorem 4.2 shows that if S is a compactly cancellative foundation semigroup with identity such that S is a union of groups, then $D\left(M_a(S)\right) \subseteq I_0(S)$.

Lemma 4.4. Let $S = \cup\{G_e : e \in E_S\}$ be a topological Clifford semigroup and $D \in Z^1\left(M(S), M(S)\right)$. If $e \in E_S$ and $\text{supp}(\mu) \subseteq G_e$, then $\text{supp}(D(\mu)) \subseteq \cup_{j \le e} G_j$.

Proof. Since e is central, so $D(\delta_e) = D(\delta_e * \delta_e) = 2\delta_e * D(\delta_e)$ and hence $\delta_e * D(\delta_e) = \delta_e * \left(2\delta_e D(\delta_e)\right) = 2\delta_e * D(\delta_e)$. Since $\text{supp}(\mu) \subseteq G_e$, we have

$$D(\mu) = D(\mu_e * \delta_e) = D(\mu_e) * \delta_e + \mu * D(\delta_e)$$
$$= D(\mu) * \delta_e.$$

Thus $\text{supp}(D(\mu)) = \text{supp}(D(\mu) * \delta_e) \subseteq Se = \bigcup_{j \le e} G_j$. □

The following theorem is indeed the main result of this paper.

Theorem 4.5. Let $S = \cup\{G_e : e \in E_S\}$ be a topological Clifford semigroup such that E_S is finite and each G_j is closed. Then $H^1\left(M(S), M(S)\right) = \{0\}$.

Proof. Let $D \in Z^1\left(M(S), M(S)\right)$. Each $e \in E_S$ defines a bounded derivation $D_e : M(G_e) \mapsto M(S)$ by $D_e(\mu_e) = D(\overline{\mu_e})$, where $\overline{\mu_e} \in M(S)$ is given by

$$\int_S f \, d\overline{\mu_e} = \int_{G_e} (f|_{G_e}) d\mu_e \qquad (f \in C_0(S)).$$

By Lemma 4.4, $D_e\left(M(G_e)\right) \subseteq M(\cup_{j \le e} G_j)$. Since each G_j is closed and E_S is finite, so each G_j is also open and hence $M(\cup_{j \le e} G_j) = \oplus_{j \le e} M(G_j)$. Thus we have

$$D_e(M(G_e)) \subseteq M(\cup_{j \le e} G_j) = \oplus_{j \le e} M(G_j).$$

Therefore we can decompose D_e across $\oplus_{j \le e} M(G_j)$ as $D_e(\mu_e) = \sum_{j \le e} D_e^j(\mu_e)$, where $D_e^j(\mu_e)$ denotes the j th projection of $D_e(\mu_e)$ on $M\left(G_j\right)$. Since $j \le e$, so $je = j$, and hence D_e^j is a derivation from $M\left(G_e\right)$ into $M\left(G_j\right)$. We call each associated derivation from $M\left(G_e\right)$ to $M\left(G_e\right)$ the *principle component* of D on G_e. By [13], if G is a locally compact group, then

$H^1\left(M(G), M(G)\right) = 0$. By using the method of Theorem 3.2 of [3], we get a bounded derivation $D^\# = D - ad_\xi$, where $\xi \in M(S)$ and $D^\#$ has zero component on each G_e $(e \in E_S)$. If $e \le u$ and $\mu_u \in M(G_e)$, then $D^\#(\delta_e * \mu_u) = \delta_e * D^\#(\mu_u)$ and $\text{supp}(\delta_e * \mu_u) \subseteq G_e . G_u \subseteq G_{eu} = G_e$. So we can apply the argument of Theorem 3.2 of [3] to obtain $D^\# = 0$. Hence D is inner. □

Example 4.6. Let $n \in \mathbb{N}$ and $T = \{1, 2, \ldots, n\}$ with the $k.l = max\{k, l\}$ $(k, l \in T)$. Suppose G is a locally compact group. Then $S = T \times G$ with the product topology and coordinatewise multiplication defines a Clifford semigroup that satisfies the hypothesis of Theorem 4.5 Therefore $H^1\left(M(S), M(S)\right) = \{0\}$.

Remark 4.7. Let S be a left zero semigroup with at least three elements. Then $S = \cup_{s \in S} \{s\}$, but

$H^1(\ell^1(S),\ell^1(S)) \neq \{0\}$ by Lemma 2.5. Therefore Theorem 4.5 is not valid in general for every semigroup S which is a union of groups.

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