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A Note on the Integrality Gap in the Nodal Interdiction Problem

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Abstract

In the maximum flow network interdiction problem, an attacker attempts to minimize the maximum flow by interdicting flow on the arcs of network. In this paper, our focus is on the nodal interdiction for network instead of the arc interdiction. Two path inequalities for the node-only interdiction problem are represented. It has been proved that the integrality gap of relaxation of the maximum flow network interdiction problem is not bounded below by a constant, even when strengthened by the path inequalities. We show that this result is also established for the nodal interdiction problem.

Keywords: Nerwork flows; Nodal interdiction; Path inequalities

Introduction

The network interdiction problem is a classical problem in the network optimization. The major idea of this problem is as follows: In the network optimization problems, for example, the maximum flow network, the shortest path, the multiple commodity network, the stochastic network, and so on, we want to optimize the objective function by considering some constraints. But, in the network interdiction problems, an attacker tries to stop this operation by attacking either arcs or nodes or both.

The topic of interdiction has been entered in majority of the network problems [2, 6, 8]. Also, the interdiction problems serve in many of real world problems, such as the conflict resolution [4], as well as controlling the infections in a hospital [5].

One of the simplest network problems is the maximum flow network problem [1]. The maximum flow network interdiction problem has been represented in [8] by interdicting arcs which is defined as follows: For a network $G=(N,A)$ with the node set N and arc set A and two positive integers R and U, is there any subset $A' \subseteq A$ so that deletion of these arcs consumes no more than R units of resource and results the maximum flow in the network which is no more than U ?

Wood in [8] proved that maximum flow network interdiction problem is a NP-complete problem even when the interdiction of an arc requires exactly one unit of resource. He has also represented an integer linear programming model for solving the maximum flow network interdiction problem which has been widely used in the literature of the interdiction problem.

Altner et al. in [3] expended the understanding of Wood's integer linear programming for the maximum flow network interdiction problem and the approximability of this problem. They presented two

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classes of the path inequalities for the maximum flow network interdiction problem. They proved that the integrality gap of the linear programming relaxation of Wood's model is not bounded below by a constant, even when strengthened by the path inequalities.

Kennedy et al. in [7] used the nodal interdiction for minimizing the maximum flow through a network. The interdiction of a node can be converted the interdiction of an arc via the method of node splitting. The node splitting replaces node i in network with two artificial nodes i' and i'' and an arc (i', i''). In this method, the size of the network increases. However, Kennedy et al. in [7] represented yet another model for the nodal interdiction.

In this paper, it is proved that the result of [3] can be also established for the node-only interdiction problem that has been introduced in [7]. For ease of reference, we use the same notations as in [3].

Some basic facts

Let $G = (N, A)$ be a directed network, where N and A are the sets of nodes and arcs, respectively. Assume that the network has a source node $s \in N$ and a sink node $t \in N$. We associate with each arc $(i, j) \in A$ a capacity u_{ij}. A cut is a partition of N into two subsets S and $\bar{S} = N \setminus S$ such that $s \in S$ and $t \in \bar{S}$. An arc (i, j) is called forward arc of the cut if $i \in S$ and $j \in \bar{S}$. As it has been stated in [7], the node-only interdiction model can be formulated as follows:

$$
\begin{aligned}
\text{(A)}\ \operatorname{Min} z = & \sum_{(i,j)\in A} u_{ij}\beta_{ij} \\
s.t.\quad & \alpha_i - \alpha_j + \beta_{ij} + \gamma_{ij} \geq 0 \quad \forall (i,j) \in A \\
& \alpha_t - \alpha_s \geq 1 \\
& \sum_{i \in N} r_i \gamma_i \leq R \\
& \gamma_{ij} = \gamma_i \qquad \forall i \in N \\
& \alpha_i \in \{0,1\} \qquad \forall i \in N \\
& \beta_{ij}, \gamma_{ij} \in \{0,1\} \qquad \forall (i,j) \in A
\end{aligned}
$$

where r_i is the interdiction cost for the node $i \in N$ and R is the interdiction budget. The decision variables are defined as follows:

$$
\alpha_i = \begin{cases} 1 & \text{if } i \in N \text{ is on the sink side of the cut} \\ 0 & \text{otherwise} \end{cases}
$$

$$
\beta_{ij} = \begin{cases} 1 & \text{if } (i,j) \in A \text{ is a forward arc} \\ & \text{and is not interdicted} \\ 0 & \text{otherwise} \end{cases}
$$

$$
\gamma_{ij} = \begin{cases} 1 & \text{if } (i,j) \in A \text{ is a forward arc} \\ & \text{and is interdicted} \\ 0 & \text{otherwise} \end{cases}
$$

$$
\gamma_i = \begin{cases} 1 & \text{if } i \in N \text{ is interdicted} \\ 0 & \text{otherwise} \end{cases}
$$

Now, we denote the relaxation of model (A) by (A.LP), that is, we have:

$$
\begin{aligned}
\text{(A.LP)}\ \operatorname{Min} z = & \sum_{(i,j)\in A} u_{ij}\beta_{ij} \\
s.t.\quad & \alpha_i - \alpha_j + \beta_{ij} + \gamma_{ij} \geq 0 \quad \forall (i,j) \in A \\
& \alpha_t - \alpha_s \geq 1 \\
& \sum_{i \in N} r_i \gamma_i \leq R \\
& \gamma_{ij} = \gamma_i \qquad \forall i \in N \\
& \alpha_i \geq 0 \qquad \forall i \in N \\
& \beta_{ij}, \gamma_{ij} \geq 0 \qquad \forall (i,j) \in A
\end{aligned}
$$

In this paper, it is assumed that $r_i = 1$ for each node $i \in N$.

Results and Discussion

Consider a network $G = (N, A)$ with the interdiction budget R. It is emphasized that the network does not contain multiple arcs. Consider a feasible solution (α, β, γ) to the model (A). Let node $u \in N$ be on the source side of the cut and P_{u-t} indicates the set of node-disjoint paths from u to t in the network $G = (N, A)$. Also, let

$$N(P_{u-t}) = \{v \in N \setminus \{u,t\} \mid \exists p \in P_{u-t}; v \in p\}$$

and $\mathcal{P}_{u-t}^{R} = \{P_{u-t} \mid |P_{u-t}| > R\}$.

When a node $v \in N(P_{u-t})$ is interdicted, we cannot traverse a path $p \in P_{u-t}$ which contains v on the path. Now, there are two cases:

1. The length of each path in P_{u-t} is at least 2. Therefore, at least R paths can be interdicted. Since $|P_{u-t}| > R$, at least $|P_{u-t}| - R$ nodes in $N(P_{u-t})$ are

not interdicted. In other words, at least $|P_{u-t}|-R$ of the nodes in $N(P_{u-t})$ have the variable γ equals to 0.

2. The length of a path in P_{u-t} is 1, that is, this path is the arc (u,t). Since the arc (u,t) is not interdicted and $|P_{u-t}|>R$, then at least $|P_{u-t}|-R$ of the nodes in $N(P_{u-t})$ have the variable γ equals to 0.

Using these two cases, we can represent the node-to-sink path inequalities as follows:

$$(|P_{u-t}|-R)\alpha_u + \sum_{i\in N(P_{u-t})}(1-\gamma_i) \geq |P_{u-t}|-R,$$

where $P_{u-t} \in \mathcal{P}_{u-t}^{R}$.

It is certain that $\alpha_u = 0$ or 1. If $\alpha_u = 1$, the node-to-sink path inequalities become $\sum_{i\in N(P_{u-t})}(1-\gamma_i) \geq 0$ that it is valid. If $\alpha_u = 0$, then according to the above two cases, the inequalities hold. The source-to-node path inequalities are similar to the node-to-sink path inequalities. It follows that, employing the same idea, the source-to-node path inequalities can be obtained as follows:

$$\sum_{i\in N(P_{s-u})}(1-\gamma_i) \geq (|P_{s-u}|-R)\alpha_u,$$

where $P_{s-u} \in \mathcal{P}_{s-u}^{R}$.

We construct an instance and obtain the optimal solution of (A) for this instance. Let k and μ be positive integers such that $\mu > 1$ and $k > \mu$. An instance $I_{\mu,k}$ of (A) is constructed as follows: The node set N and the arc set A are partitioned as follows:

$$N = \{s,t\} \cup X \cup Z, \quad |X| = 2k, \quad |Z| = \mu$$

$$A = A_{X_1} \cup A_{X_2} \cup A_{Z_1} \cup A_{Z_2}$$

$$\begin{cases} A_{X_1} = \{(s,v) \mid v \in X\} \\ A_{X_2} = \{(v,t) \mid v \in X\} \\ A_{Z_1} = \{(s,v) \mid v \in Z\} \\ A_{Z_2} = \{(v,t) \mid v \in Z\} \end{cases}$$

The arc capacities are defined as follows:

$$u_{ij} = \begin{cases} 1 & (i,j) \in A_{X_1} \\ \mu & (i,j) \in A_{X_2} \\ 1 & (i,j) \in A_{Z_1} \\ \mu & (i,j) \in A_{Z_2} \end{cases}$$

The interdiction budget is $R = \mu + k - 1$.

Now, the optimal objective value of (A) is obtained for the instance $I_{\mu,k}$ as follows:

Theorem 1. The optimal objective value of (A) for the instance $I_{\mu,k}$ is $k+1$.

Proof. Consider a solution in which the variables γ and α are defined in the following way:

$$\alpha_i = \begin{cases} 0 & \text{if } i \in X^* \cup Z^* \cup \{s\} \\ 1 & \text{otherwise} \end{cases}$$

$$\gamma_i = \begin{cases} 1 & \text{if } i \in X^* \cup Z^* \\ 0 & \text{otherwise} \end{cases}$$

where $X^* \subseteq X$ and $Z^* \subseteq Z$ with $|X^*| = k$ and $|Z^*| = \mu - 1$. Therefore, the variables γ_{ij} and β_{ij}, for each $(i,j) \in E$, are defined as follows:

$$\gamma_{ij} = \begin{cases} 1 & \text{if } (i,j) \in X^* \times \{t\} \text{ or } (i,j) \in Z^* \times \{t\} \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \{s\} \times X \setminus X^* \\ & \text{or } (i,j) \in \{s\} \times Z \setminus Z^* \\ 0 & \text{otherwise} \end{cases}$$

It is straightforward to show that this solution is feasible for (A). The objective value of this solution is $k+1$, since:

$$z = \sum_{(i,j)\in A} u_{ij}\beta_{ij} = |X \setminus X^*| + |Z \setminus Z^*| = k+1.$$

In this solution, k nodes in X and $\mu - 1$ nodes in Z are interdicted. Now, by contrary, assume that there exists a feasible solution with the objective value strictly less than $b = k+1$, that is, $\sum_{(i,j)\in A} u_{ij}\beta_{ij} < b$. Since $\beta_{ij} = 0$ or 1, at most $b-1$ arcs in this solution must have their corresponding variable β equals to 1. in other words, at most $b-1$ nodes are not interdict. Then, we have:

$$k+\mu+2 \le \sum_{i\in N}\gamma_i \le R = \mu+k-1,$$

which is a contradiction. ■

We denote the model (A.LP) along with the path inequalities as (S.LP) and obtain a feasible solution for it.

$$
\begin{aligned}
\text{(S.LP)}\ \text{Min}\, z = & \sum_{(i,j)\in A} u_{ij}\beta_{ij} \\
s.t.\ \ & \alpha_i - \alpha_j + \beta_{ij} + \gamma_{ij} \ge 0 \quad \forall (i,j)\in A \\
& \alpha_t - \alpha_s \ge 1 \\
& \sum_{i\in N} r_i\,\gamma_i \le R \\
& \gamma_{ij} = \gamma_i \qquad \forall i \in N \\
& (|P_{u-t}| - R)\alpha_u + \sum_{i\in N(P_{u-t})}(1-\gamma_i) \ge |P_{u-t}| - R \\
& \sum_{i\in N(P_{s-u})}(1-\gamma_i) \ge (|P_{s-u}| - R)\alpha_u \\
& \alpha_i \ge 0 \qquad \forall i\in N \\
& \beta_{ij}, \gamma_{ij} \ge 0 \qquad \forall (i,j)\in A \\
& P_{u-t} \in \mathcal{P}^R_{u-t},\ P_{s-u} \in \mathcal{P}^R_{s-u}
\end{aligned}
$$

Theorem 2. There exists a feasible solution for the model (S.LP) of instance $I_{\mu,k}$ that has the objective value of $1+2\frac{k}{\mu}$.

Proof. Consider the following solution:

$$\alpha_i = \begin{cases} 0 & i\in\{s\}\cup\overline{X} \\ \dfrac{1}{\mu} & i\in Z\cup\overline{X}' \\ 1 & i=t \end{cases}$$

$$\beta_{ij} = \begin{cases} 0 & (i,j)\in X\times\{t\}\ \text{or}\ (i,j)\in Z\times\{t\} \\ \dfrac{1}{\mu} & (i,j)\in\{s\}\times X\ \text{or}\ (i,j)\in\{s\}\times Z \end{cases}$$

$$\gamma_i = \begin{cases} 0 & i\in\{s,t\}\cup\overline{X}' \\ 1 & Z\cup\overline{X} \end{cases}$$

where $\overline{X}\subseteq X, |\overline{X}| = k-1$ and $\overline{X}' = X\setminus\overline{X}$. It is straightforward to see that this solution is feasible for (S.LP). The objective value of this solution is $1+2\frac{k}{\mu}$. ■

For an instance I, let $Z^*_{\text{A}}(I)$ denotes the optimal objective value of (A), $Z^*_{\text{A.LP}}(I)$ denotes the optimal objective value of (A.LP) and $Z^*_{\text{S.LP}}(I)$ denotes the optimal objective value of (S.LP). We now prove the following theorem:

Theorem 3. Let $\varepsilon\in(0,1)$ and $|N| = n$. For all sufficiently large positive integers n, there exists an instance I of (A) such that

1. $\dfrac{Z^*_{\text{A}}(I)}{Z^*_{\text{S.LP}}(I)} \in \Omega(n^{1-\varepsilon})$,
2. $\dfrac{Z^*_{\text{A}}(I)}{Z^*_{\text{A.LP}}(I)} \in \Omega(n^{1-\varepsilon})$.

Proof. Consider the instance $I = I_{\mu,k}$. According to Theorems 1 and 2, we have:

$$\frac{Z^*_{\text{A}}(I_{\mu,k})}{Z^*_{\text{S.LP}}(I_{\mu,k})} \ge \frac{k+1}{1+2\frac{k}{\mu}}.$$

This inequality together with $n = 2k+\mu+2$ imply that

$$\frac{Z^*_{\text{A}}(I_{\mu,k})}{Z^*_{\text{S.LP}}(I_{\mu,k})} \ge \frac{\mu(n-\mu)}{2(n-2)}.$$

For given $\varepsilon\in(0,1)$, we choose $\mu = n^{1-\varepsilon}$. Therefore, we have

$$\frac{Z^*_{\text{A}}(I_{\mu,k})}{Z^*_{\text{S.LP}}(I_{\mu,k})} \ge \frac{n^{1-\varepsilon}(n-n^{1-\varepsilon})}{2(n-2)},$$

which proves the first part of the theorem. Now, using the fact that

$$Z^*_{\text{S.LP}}(I_{\mu,k}) \ge Z^*_{\text{A.LP}}(I_{\mu,k}),$$

one can easily conclude the second part of the theorem by using the first part. ■

By constructing an instance, it was proved that $\dfrac{Z^*_{\text{A}}(I)}{Z^*_{\text{S.LP}}(I)} \in \Omega(n^{1-\varepsilon})$. This relation shows that the integrality gap of relaxation of the nodal interdiction problem along with the path inequalities is not bounded below by a constant.

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References

1. Ahuja R. K., Magnanti T. L., and Orlin J. B. Network Flows: Theory, Algorithms and Applications. Prentice Hall, Upper Saddle River, New Jersey (1993).
2. Akgun I., Tansel B. C., and Wood R. K. The multi-terminal maximum-flow network-interdiction problem. *EJOR*, **211**(2): 241-251 (2011).
3. Altner D. S., Ergun O., and Uhan N. A. The maximum flow network interdiction problem: valid inequalities, integrality gaps, and approximability. *Oper. Res. Lett.*, **38**(1): 33-38 (2010).
4. Anandalingam G., and Apprey V. Multi-level programming and conflict resolution. *EJOR*, **51**(2): 233-247 (1991).
5. Assimakopoulos N. A network interdiction model for hospital infection control. *Comput. Biol. Med.*, **17**(6): 413-422 (1987).
6. Israeli E., and Wood R. K. Shortest path network interdiction. *Networks*, **40**(2): 97-111 (2002).
7. Kennedy K. T., Deckro R. F., Moore J. T., and Hopkinson K. M. Nodal interdiction. *Math. Comput. Modelling*, **54**(11-12): 3116-3125 (2011).
8. Wood R. K. Deterministic network interdiction. *Math. Comput. Modelling*, **17**(2): 1-18 (1993).