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Modified F-Expansion Method Applied to Coupled System of Equation

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Abstract

A modified F-expansion method to find the exact traveling wave solutions of two-component nonlinear partial differential equations (NLPDEs) is discussed. We use this method to construct many new solutions to the nonlinear Whitham-Broer-Kaup system (1+1)-dimensional. The solutions obtained include Jacobi elliptic periodic wave solutions which exactly degenerate to the soliton solutions, triangular periodic wave solutions, exponential solutions and rational solutions under certain limitet conditions. In addition, some figures of partial solutions are provided for direct-viewing analysis. The method can also be extended to other types of nonlinear evolution equations in mathematical physics.

Keywords: Modified F-expansion Method, Exact Solution, Whitham-Broer-Kaup System (1+1)-dimensional

Introduction

The discovery of the soliton, its remarkable properties and the incredible richness of structure are all included in its mathematical description. The story begins with the observation by John Scott Russell of "the great wave of translation". He states:

"…Its height gradually diminished, and after a chase of one or two miles 1 lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation, a name which it now very generally bears [11]…"

The appearance of solitary wave solutions in nature is quite common. Refereces can be made to Bell-shaped sech-solutions and kink-shaped tanh-solutions model wave phenomena in fluids, plasmas, elastic media, electrical circuits, optical fibers, chemical reactions, bio-genetics, etc. The travelling wave solutions of the Korteweg–de Vries (KdV) and Boussinesq equations, which describe water waves, are famous examples as well.

In recent years, other methods have been developed, such as the Backlund transformation method [9], Darboux transformation [8], tanh method [10, 4], extended tanh function method [5], Exp-function method [3, 7], the generalized hyperbolic function [6], the variable separation method [10], $(\frac{G'}{G})$ method [2] and the extended Jacobi elliptic function expansion method[1]. All the above-mentioned approaches are

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based on the assumption that the solutions can be uniformly expressed in terms of some special ansatz. Therefore, the original partial differential equations (PDE's) can be transformed into a set of algebraic equations through balancing the same order of the ansatz, which yields the explicit expressions of the waves. The difference between these methods is attributed to the different ansatz introduced. For example, in the tanh-coth method, the ansatz can be written in as combinations of tanh and coth functions, while in the Jacobi elliptic function expansion method, the ansatz can be expressed in the form of Jacobi elliptic functions.

From our point of view, all these methods have some merits and demerits with respect to the problem considered and there is no unified method that can be used to deal with all types of NLPDEs. That is why anytime that an improvement is made in a particular method to allow it to recover some new solutions to the NLPDEs, it is always welcomed. The purpose of this paper is to apply a modified F-expansion method to coupled families of NLPDEs.

The aim of this paper is organized as follows: In Section 2,at first, we briefly present the steps of the modified F-expansion method. In Section 3, by using the results obtained in Section 2, attempts are made to apply the method to solve the Whitham-Broer-Kaup system (1+1)-dimensional.

1. Description of Method

Consider a general nonlinear partial differential equation with independent variables $x=(t,x_1,x_2,\dots,x_m)$ and dependent variables u, in the form:

$$F\big(u,u_t,u_{x_1},u_{x_2},\dots,u_{x_m},u_{tx_1},u_{tx_2},\dots,u_{tx_m},u_{tt},u_{x_1x_1},u_{x_2x_2},\dots,u_{x_mx_m},\dots\big)=0,\quad (1\text{-}1)$$

where $u=u(x,t)$ is the solution of nonlinear PDE Eq.(1-1). Furthermore, the transformations which are used are as follows:

$$u(x_1,x_2,\dots,x_m,t)=U(\xi),\ \xi=k_1(x_1+k_2x_2+\cdots+k_mx_m-\lambda t).\quad (1\text{-}2)$$

where λ and k_i are constants. Using the chain rule, it can be found that:

$$\frac{\partial}{\partial t}(.)=-k_1\lambda\frac{\partial}{\partial\xi}(.),\ \ \frac{\partial}{\partial x_1}(.)=k_1\frac{\partial}{\partial\xi}(.),\ \ \frac{\partial}{\partial x_2}(.)=k_1k_2\frac{\partial}{\partial\xi}(.),\dots\quad (1\text{-}3)$$

At present, Eq.(1-3) is employed to change the nonlinear PDE Eq.(1-1) to nonlinear ordinary differential equation:

$$G\big(U(\xi),U_\xi(\xi),U_{\xi\xi}(\xi),\dots\big)=0.\quad (1\text{-}4)$$

According to the modified F-expansion method, it is assumed that the solution can be expressed in the form:

$$U(\xi)=a_0+\sum_{i=1}^{N}a_iF^i(\xi)+\sum_{i=1}^{N}b_iF^{-i}(\xi)\quad (1-5)$$

where $a_0,a_i$and b_i are constants to be determined. $F(\xi)$ satisfies Riccati equation:

$$F'(\xi)=A+BF(\xi)+CF^2(\xi).\quad (1\text{-}6)$$

whereA,B and C are constants to be determined. The prime $'$denotes $d/d\xi$. Integer Ncan be determined by considering the homogeneous balance between the governing nonlinear term(s) and highest order derivatives of $U(\xi)$ in Eq.(1-4). Given different values of $A,B$and C, the different Riccati function solution, $F(\xi)$ can be obtained from Eq.(1-6) (see Table 1). To determine $U(\xi)$ explicitly, we take the following steps:

Step I. Substituting (1-5) along with (1-6) into Eq.(1-4) and collect coefficients of $F^i(\xi)$ $(i=-N,\dots,N)$, then set each coefficient to zero. Equating each coefficient of $F^i(\xi)$ to zero yields a system of algebraic equations for $a_i(i=N,\dots,1,0),b_i(i=1,\dots,N),k_i(i=1,\dots,m)$ and λ.

Step II. Solving the system of algebraic equations, probably with the aid of *Mathematica*or *Maple*. a_i $(i=N,\dots,1,0)$ and b_i $(i=N,\dots,1)$ can be expressed by A,B and C (or the coefficients of ODE(1-4). Substituting these results into (1-5), we can obtain the general form of travelling wave solutions to Eq.(1-4).

Step III. Selecting A,B,C and $F(\xi)$ from Table 1 and substituting them along with a_i $(i=N,\dots,1,0)$ and b_i $(i=N,\dots,1)$ into Eq.(1-5), a series of soliton-like solutions, trigonometric function solutions and rational solutions to Eq.(1-4) can be obtained.

The modified F-expansion method is more effective in obtaining the soliton-like solution, trigonometric function solutions, exponential solutions and rational solutions of the nonlinear partial deferential equations. This method will yield more rich solutions types of the nonlinear partial deferential equations. It shows that the modified F-expansion method is more powerful in constructing exact solutions of NLPDEs.

Relations between values of A,B,C and corresponding $F(\xi)$ in Eq.(6) are listed in (Table 1).

Table 1. Relations between values of A, B, C and corresponding F(ξ) in Eq.(6)

Values of A, B, C	F(ξ)
$A=0, B=1, C=-1$	$\frac{1}{2}+\frac{1}{2}\tanh(\frac{1}{2}\xi)$
$A=0, B=-1, C=1$	$\frac{1}{2}-\frac{1}{2}\coth(\frac{1}{2}\xi)$
$A=\frac{1}{2}, B=0, C=-\frac{1}{2}$	$\coth(\xi)\pm\mathrm{csch}(\xi)$, $\tanh(\xi)\pm i\,\mathrm{sech}(\xi)$
$A=1, B=0, C=-1$	$\tanh(\xi), \coth(\xi)$
$A=\frac{1}{2}, B=0, C=\frac{1}{2}$	$\sec(\xi)+\tan(\xi)$, $\csc(\xi)-\cot(\xi)$
$A=-\frac{1}{2}, B=0, C=-\frac{1}{2}$	$\sec(\xi)-\tan(\xi)$, $\csc(\xi)+\cot(\xi)$
$A=1(-1), B=0, C=1(-1)$	$\tan(\xi)\ (\cot(\xi))$
$A=0, B=0, C\neq 0$	$-\frac{1}{C\xi+\eta}$(η is anarbitrary constant)
A is arbitaray constant, $B=0, C=0$	$A\xi$
A is arbitaray constant, $B\neq 0, C=0$	$\frac{\exp(B\xi)-A}{B}$

Results

2. Exact solutions to Whitham-Broer-Kaup coupled system of equation

The Whitham-Broer-Kaup (WBK) equation is written in the form:

$$u_t+uu_x+v_x+\beta u_{xx}=0,$$
$$v_t+(vu)_x+\alpha u_{xxx}-\beta v_{xx}=0. \quad (2\text{-}7)$$

where α and β are real constants. Due to the convective, dispersive and viscous effects [13], the WBK equation is a valuable model of long waves.When specific values are taken for the parameters α and β, system (2-7) will be transformed into many important mathematical and physical equations. For example, when $\alpha=0, \beta\neq 0$, Eq. (2-7) becomes a classical long wave equation describing shallow water with dispersive, whereas if $\alpha=1, \beta=0$, the system becomes a variant Boussinesq equation. Furthermore, many important models are extensions of the WBK equation, such as the generalized Broer-Kaup equation [12].

By using the transformation:

$$u(x,t)=U(\xi),\quad v(x,t)=V(\xi),\ \text{and}\ \xi=x+\lambda t. \quad (2\text{-}8)$$

where λ is arbitrary constant, and substituting Eq.(2-8) with Eq.(2-7), there will be a change as follows:

$$\lambda U_\xi(\xi)+U(\xi)U_\xi(\xi)+V_\xi(\xi)+\beta U_{\xi\xi}(\xi)=0,$$
$$\lambda V_\xi(\xi)+\left(U(\xi)V(\xi)\right)_\xi+\alpha U_{\xi\xi\xi}(\xi)-\beta V_{\xi\xi}(\xi)=0. \quad (2\text{-}9)$$

where by integrating the first equation of the Eq.(2-9) with respect to ξ, it can be found that:

$$V(\xi)=c_1-\frac{1}{2}\left(U^2(\xi)+2\lambda U(\xi)+2\beta U_\xi(\xi)\right). \quad (2\text{-}10)$$

where c_1 is an integral constant. Substituting Eq.(2-10) with the second equation of the second Eq.(2-9) results in:

$$(2c_1-2\lambda^2)U_\xi(\xi)-6\lambda U(\xi)U_\xi(\xi)-3U^2(\xi)U_\xi(\xi)+2(\alpha+\beta^2)U_{\xi\xi\xi}(\xi)=0. \quad (2\text{-}11)$$

Integrating Eq.(2-11) once yields:

$$(2c_1-2\lambda^2)U(\xi)-3\lambda U^2(\xi)-U^3(\xi)+2(\alpha+\beta^2)U_{\xi\xi}(\xi)-c_2=0. \quad (2\text{-}12)$$

where c_2 is an integral constant. Considering the homogeneous balance between U^3 and $U_{\xi\xi}$ in (2-12), we suppose that the solution to ordinary differential equation (2-12) can be expressed by

$$U(\xi)=a_0+a_1F(\xi)+b_1F^{-1}(\xi). \quad (2\text{-}13)$$

Where a_0, a_1 and b_1 are constants to be determined. Substituting (2-13) with Eq.(2-12), and using (1-6), the left-hand side of Eq.(2-13) can be converted into a finite series in $F^j(\xi)(j=-3,\ldots,3)$. Equating each coefficient of $F^j(\xi)$ to zero yields a system of algebraic equations for a_0, a_1, b_1, c_1 and c_2:

$F^3:\ -a_1^3+4a_1C^2(\alpha+\beta^2)=0,$
$F^2:\ -3\lambda a_1^2-3a_0a_1^2+6a_1BC(\alpha+\beta^2)=0,$
$F^1:\ 2(c_1-\lambda^2)a_1-6a_0a_1\lambda-3(a_1b_1+a_0^2)a_1+2a_1(\alpha+\beta^2)(B^2+2AC)=0,$
$F^0:\ 2(c_1-\lambda^2)a_0-3\lambda(2a_1b_1+a_0^2)-(a_0^3+6a_0a_1b_1)+2(\alpha+\beta^2)\times\times(BCb_1+ABa_1)-c_2=0,$ (2-14)
$F^{-1}:2(c_1-\lambda^2)b_1-6a_0b_1\lambda-3(a_1b_1+a_0^2)b_1+2(\alpha+\beta^2)(B^2+2A)b_1=0,$
$F^{-2}:\ -3(a_0+\lambda)b_1^2+6ABb_1(\alpha+\beta^2)=0,$
$F^{-3}:\ -b_1^3+4A^2b_1(\alpha+\beta^2)=0.$

Solving the algebraic equations (2-14) using *Maple*, the following solutions will be obtained:

Case I:
$a_0=B\sqrt{\alpha+\beta^2}-\lambda, a_1=2C\sqrt{\alpha+\beta^2}, b_1=0, c_1=(\frac{1}{2}B^2-2AC)(\alpha+\beta^2)-\frac{1}{2}\lambda^2,$
$c_2=2\lambda(B^2+2AC)(\alpha+\beta^2)+\lambda^2\left(4\lambda-6B\sqrt{\alpha+\beta^2}\right)-3\lambda a_0^2.$ (2-15)

Case II:
$a_0=B\sqrt{\alpha+\beta^2}-1, a_1=0, b_1=2A\sqrt{\alpha+\beta^2}, c_1=(\frac{1}{2}B^2-2AC)(\alpha+\beta^2)-\frac{1}{2}\lambda^2,$
$c_2=2\lambda(B^2+2AC)(\alpha+\beta^2)+\lambda^2\left(4\lambda-6B\sqrt{\alpha+\beta^2}\right)-3\lambda a_0^2.$ (2-16)

Case III:
$a_0=B\sqrt{\alpha+\beta^2}-\lambda, a_1=2A\sqrt{\alpha+\beta^2}, b_1=2C\sqrt{\alpha+\beta^2}, c_1=\left(\frac{1}{2}B^2-2AC\right)\times(\alpha+\beta^2)-\frac{1}{2}\lambda^2,\ c_2=2\lambda(B^2+2AC)(\alpha+\beta^2)+\lambda^2\left(4\lambda-6B\sqrt{\alpha+\beta^2}\right)-3\lambda a_0^2.$ (2-17)

Substituting (2-15), (2-16) and (2-17) with Eq.(2-13), from Table 1, we may obtain many soliton-like solutions, trigonometric function solutions, exponential solutions and rational solutions to Eq.(2-7) (where we left the same type solutions out):

2-1. The soliton-like solutions to Whitham-Broer-Kaup System

(1) When $A=0, B=1, C=-1$, from Table 1, $F(\xi)=\frac{1}{2}+\frac{1}{2}\tanh(\frac{1}{2}\xi)$. By *case* I, the exact solution to equation (2-7) is given by:
$u_1(x,t)=-\lambda-\sqrt{\alpha+\beta^2}\tanh(\frac{1}{2}(x+\lambda t)).$
$v_1(x,t)=\frac{1}{2}(\alpha+\beta^2-\lambda^2)-\frac{1}{2}[u_1^2(x,t)+2\lambda u_1(x,t)-2\beta\sqrt{\alpha+\beta^2}(1-\tanh^2(\frac{1}{2}(x+\lambda t))].$

(2) When $A=0, B=-1, C=1$, from Table 1, $F(\xi)=\frac{1}{2}-\frac{1}{2}\coth(\frac{1}{2}\xi)$. By *case* I, the exact solution to equation (2-7) is given by:
$u_2(x,t)=-\lambda-\sqrt{\alpha+\beta^2}\coth(\frac{1}{2}(x+\lambda t)).$
$v_2(x,t)=-\frac{1}{2}(\alpha+\beta^2+\lambda^2)-\frac{1}{2}[u_2^2(x,t)+2\lambda u_2(x,t)-2\beta\sqrt{\alpha+\beta^2}(1-\coth^2(\frac{1}{2}(x+\lambda t))].$

(3) When $A=\frac{1}{2}, B=0, C=-\frac{1}{2}$, from Table 1, $F(\xi)=\coth(\xi)\pm\operatorname{csch}(\xi)\ or\ \tanh(\xi)\pm i\operatorname{sech}(\xi)$. By *case* I, the exact solution to equation (2-7) is given by:
$u_3(x,t)=-\lambda-\sqrt{\alpha+\beta^2}\,[\coth(x+\lambda t)\pm csch(x+\lambda t)].$
$v_3(x,t)=\frac{1}{2}(\alpha+\beta^2)-\frac{1}{2}[u_3^2(x,t)+2\lambda u_3(x,t)-2\beta\sqrt{\alpha+\beta^2}(1-\coth^2(x+\lambda t)\mp csch(x+\lambda t)coth(x+\lambda t))].$
$u_4(x,t)=-\lambda-\sqrt{\alpha+\beta^2}\,[\tanh(x+\lambda t)\pm i\,sech(x+\lambda t)].$
$v_4(x,t)=\frac{1}{2}(\alpha+\beta^2)-\frac{1}{2}[u_4^2(x,t)+2\lambda u_4(x,t)-2\beta\sqrt{\alpha+\beta^2}(1-\tanh^2(x+\lambda t)\mp i\,sech(x+\lambda t)tanh(x+\lambda t))].$

By *case* II, the exact solution to equation (2-7) can be written as:
$u_5(x,t)=-1+\sqrt{\alpha+\beta^2}[coth(x+\lambda t)\pm csch(x+\lambda t)]^{-1}.$
$v_5(x,t)=\frac{1}{2}(\alpha+\beta^2)-\frac{1}{2}[u_5^2(x,t)+2\lambda u_5(x,t)-2\beta\sqrt{\alpha+\beta^2}(\frac{1-\coth^2(x+\lambda t)\mp csch(x+\lambda t)coth(x+\lambda t)}{(\coth(x+\lambda t)\pm csch(x+\lambda t))^2})].$
$u_6(x,t)=-1+\sqrt{\alpha+\beta^2}[\tanh(x+\lambda t)\pm i\,sech(x+\lambda t)]^{-1}.$
$v_6(x,t)=\frac{1}{2}(\alpha+\beta^2)-\frac{1}{2}[u_6^2(x,t)+2\lambda u_6(x,t)-2\beta\sqrt{\alpha+\beta^2}(\frac{1-\tanh^2(x+\lambda t)\mp i\,sech(x+\lambda t)tanh(x+\lambda t)}{(\tanh(x+\lambda t)\pm i\,sech(x+\lambda t))^2})].$

By *case* III, the exact solution to equation (2-7) will be shown as follows:
$u_7(x,t)=-\lambda-\sqrt{\alpha+\beta^2}[\coth(x+\lambda t)\pm csch(x+\lambda t)]+\sqrt{\alpha+\beta^2}[\coth(x+\lambda t)\pm csch(x+\lambda t)]^{-1}.$
$v_7(x,t)=\frac{1}{2}(\alpha+\beta^2)-\frac{1}{2}[u_7^2(x,t)+2\lambda u_7(x,t)-2\beta\sqrt{\alpha+\beta^2}(1-\coth^2(x+\lambda t)\mp csch(x+\lambda t)coth(x+\lambda t))-$

$2\beta\sqrt{\alpha+\beta^2}(\frac{1-\coth^2(x+\lambda t)\mp csch(x+\lambda t)coth(x+\lambda t)}{(\coth(x+\lambda t)\pm csch(x+\lambda t))^2})]$.

$u_8(x,t)=-\lambda-\sqrt{\alpha+\beta^2}[\tanh(x+\lambda t)\pm i\,sech(x+\lambda t)]+\sqrt{\alpha+\beta^2}[\tanh(x+\lambda t)\pm i\,sech(x+\lambda t)]^{-1}$.

$v_8(x,t)=\frac{1}{2}(\alpha+\beta^2)-\frac{1}{2}[u_8^2(x,t)+2\lambda u_8(x,t)-2\beta\sqrt{\alpha+\beta^2}(1-\tanh^2(x+\lambda t)\mp i\,sech(x+\lambda t)tanh(x+\lambda t))-2\beta\sqrt{\alpha+\beta^2}(\frac{1-\tanh^2(x+\lambda t)\mp i\,sech(x+\lambda t)tanh(x+\lambda t)}{(\tanh(x+\lambda t)\pm i\,sech(x+\lambda t))^2})]$.

(4) When $A=1,B=0,C=-1$, from Table 1, $F(\xi)=\tanh(\xi)\ or\ \coth(\xi)$. By *case* II, the exact solution to equation (2-7) can be written as:

$u_9(x,t)=-1+2\sqrt{\alpha+\beta^2}\coth(x+\lambda t)$.

$v_9(x,t)=2(\alpha+\beta^2)-\frac{1}{2}\lambda^2-\frac{1}{2}[u_9^2(x,t)+2\lambda u_9(x,t)+2\beta\sqrt{\alpha+\beta^2}(1-\coth^2(x+\lambda t)]$.

$u_{10}(x,t)=-1+2\sqrt{\alpha+\beta^2}\tanh(x+\lambda t)$.

$v_{10}(x,t)=2(\alpha+\beta^2)-\frac{1}{2}\lambda^2-\frac{1}{2}[u_{10}^2(x,t)+2\lambda u_{10}(x,t)+2\beta\sqrt{\alpha+\beta^2}(1-\tanh^2(x+\lambda t)]$.

By *case* III, the exact solution to equation (2-7) is given by:

$u_{11}(x,t)=-\lambda+2\sqrt{\alpha+\beta^2}\,[\coth(x+\lambda t)-\tanh(x+\lambda t)]$.

$v_{11}(x,t)=2(\alpha+\beta^2)-\frac{1}{2}\lambda^2-\frac{1}{2}[u_{11}^2(x,t)+2\lambda u_{11}(x,t)+2\beta\sqrt{\alpha+\beta^2}(\tanh^2(x+\lambda t)-\coth^2(x+\lambda t))]$.

$u_{12}(x,t)=-\lambda+2\sqrt{\alpha+\beta^2}\,[\tanh(x+\lambda t)-\coth(x+\lambda t)\,]$.

$v_{12}(x,t)=2(\alpha+\beta^2)-\frac{1}{2}\lambda^2-\frac{1}{2}[u_{12}^2(x,t)+2\lambda u_{12}(x,t)+2\beta\sqrt{\alpha+\beta^2}(\coth^2(x+\lambda t)-\tanh^2(x+\lambda t))]$.

For direct-viewing analysis, we provide the figures of $u_5(x,t)$, where we choose $\alpha=2,\beta^2=2$ and $\lambda=1$.

2-2. The trigonometric function solutions to Whitham-Broer-Kaup system

(1) When $A=\frac{1}{2},B=0,C=\frac{1}{2}$, from Table 1, $F(\xi)=\sec(\xi)+\tan(\xi)\ or\ \csc(\xi)-\cot(\xi)$. By *case* I, the exact solution to equation (2-7) is given by:

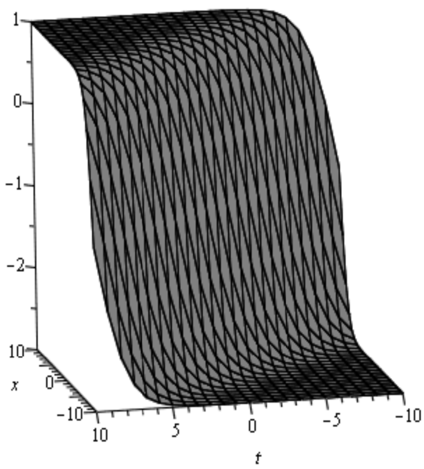
$u_{13}(x,t)=-\lambda+\sqrt{\alpha+\beta^2}\,[\sec(x+\lambda t)+\tan(x+\lambda t)\,]$,

$v_{13}(x,t)=-\frac{1}{2}(\alpha+\beta^2+\lambda^2)-\frac{1}{2}[u_{13}^2(x,t)+2\lambda u_{13}(x,t)+2\beta\sqrt{\alpha+\beta^2}(1+\tan^2(x+\lambda t)+\sec(x+\lambda t)\tan(x+\lambda t))]$.

$u_{14}(x,t)=-\lambda+\sqrt{\alpha+\beta^2}\,[\csc(x+\lambda t)-\cot(x+\lambda t)\,]$.

$v_{14}(x,t)=-\frac{1}{2}(\alpha+\beta^2+\lambda^2)-\frac{1}{2}[u_{14}^2(x,t)+2\lambda u_{14}(x,t)+2\beta\sqrt{\alpha+\beta^2}(-1+\cot^2(x+\lambda t)-\csc(x+\lambda t)\cot(x+\lambda t))]$.

By *case* II, the exact solution to equation (2-7) can be written as:



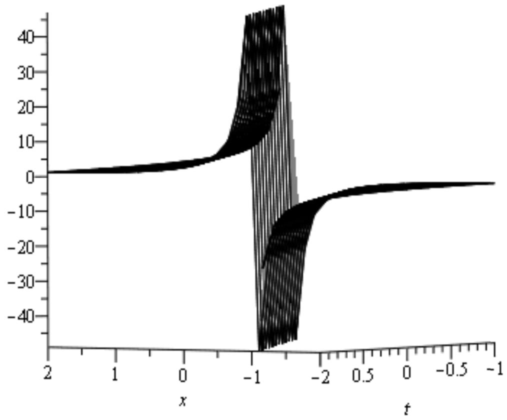


Figure 1. Graphics of soliton-like solution $u_5$are shown at "+" and "-", respectively

$u_{15}(x,t) = -1 + \sqrt{\alpha+\beta^2}\,[\sec(x+\lambda t) + \tan(x+\lambda t)]^{-1}$.
$v_{15}(x,t) = -\frac{1}{2}(\alpha+\beta^2+\lambda^2) - \frac{1}{2}[u_{15}^2(x,t) + 2\lambda u_{15}(x,t) - 2\beta\sqrt{\alpha+\beta^2}(\frac{1+\tan^2(x+\lambda t)+ sec(x+\lambda t)tan(x+\lambda t)}{(\tan(x+\lambda t)+sec(x+\lambda t))^2})]$.

$u_{16}(x,t) = -1 + \sqrt{\alpha+\beta^2}\,[\csc(\frac{1}{2}(x+\lambda t)) - \cot(\frac{1}{2}(x+\lambda t))]^{-1}$.
$v_{16}(x,t) = -\frac{1}{2}(\alpha+\beta^2+\lambda^2) - \frac{1}{2}[u_{16}^2(x,t) + 2\lambda u_{16}(x,t) - 2\beta\sqrt{\alpha+\beta^2}(\frac{-1+\cot^2(x+\lambda t)-csc(x+\lambda t)cot(x+\lambda t)}{(\csc(x+\lambda t)-cot(x+\lambda t))^2})]$.

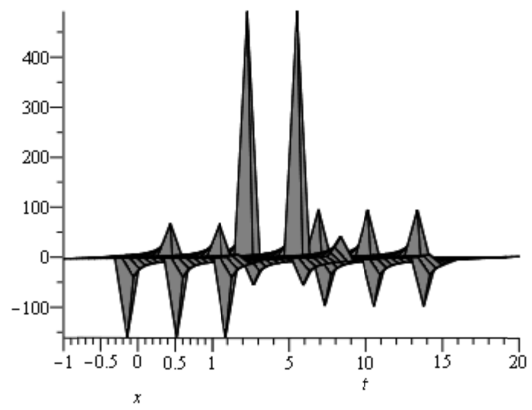
(2) When $A=-\frac{1}{2}, B=0, C=-\frac{1}{2}$, from Table 1, $F(\xi)=\sec(\xi)-\tan(\xi)\; or\; \csc(\xi)+\cot(\xi)$. By *case* I, the exact solution to equation (2-7) will be shown as follows:
$u_{17}(x,t) = -\lambda - \sqrt{\alpha+\beta^2}\,[\sec(x+\lambda t) - \tan(x+\lambda t)\,]$.
$v_{17}(x,t) = -\frac{1}{2}(\alpha+\beta^2+\lambda^2) - \frac{1}{2}[u_{17}^2(x,t) + 2\lambda u_{17}(x,t) - 2\beta\sqrt{\alpha+\beta^2}(-1-\tan^2(x+\lambda t)+ sec(x+\lambda t)tan(x+\lambda t))]$.
$u_{18}(x,t) = -\lambda - \sqrt{\alpha+\beta^2}\,[\csc(x+\lambda t) + \cot(x+\lambda t)\,]$.
$v_{18}(x,t) = -\frac{1}{2}(\alpha+\beta^2+\lambda^2) - \frac{1}{2}[u_{18}^2(x,t) + 2\lambda u_{18}(x,t) - 2\beta\sqrt{\alpha+\beta^2}(1-\cot^2(x+\lambda t) - csc(x+\lambda t)cot(x+\lambda t))]$

By *case* II, the exact solution to equation (2-7) can be written as:
$u_{19}(x,t) = -1 - \sqrt{\alpha+\beta^2}\,[\sec(x+\lambda t) - \tan(x+\lambda t)]^{-1}$,
$v_{15}(x,t) = -\frac{1}{2}(\alpha+\beta^2+\lambda^2) - \frac{1}{2}[u_{19}^2(x,t) + 2\lambda u_{19}(x,t) + 2\beta\sqrt{\alpha+\beta^2}(\frac{-1-\tan^2(x+\lambda t)+ sec(x+\lambda t)tan(x+\lambda t)}{(\sec(x+\lambda t)-tan(x+\lambda t))^2})]$.

$u_{20}(x,t) = -1 - \sqrt{\alpha+\beta^2}\,[\csc(x+\lambda t) + \cot(x+\lambda t)]^{-1}$,
$v_{20}(x,t) = -\frac{1}{2}(\alpha+\beta^2+\lambda^2) - \frac{1}{2}[u_{20}^2(x,t) + 2\lambda u_{20}(x,t) + 2\beta\sqrt{\alpha+\beta^2}(\frac{1-\cot^2(x+\lambda t)-csc(x+\lambda t)cot(x+\lambda t)}{(\csc(x+\lambda t)+cot(x+\lambda t))^2})]$.

(3) When $A=1, B=0, C=1$, from Table 1, $F(\xi)=\tan(\xi)$. By *case* II, the exact solution to equation (2-7) is found to be:
$u_{21}(x,t) = -1 + 2\sqrt{\alpha+\beta^2}\cot(x+\lambda t)$.
$v_{21}(x,t) = -2(\alpha+\beta^2) - \frac{1}{2}\lambda^2 - \frac{1}{2}[u_{21}^2(x,t) + 2\lambda u_{21}(x,t) - 4\beta\sqrt{\alpha+\beta^2}(1+\cot^2(x+\lambda t))]$.

By *case* III, the exact solution to equation (2-7) is given by:
$u_{24}(x,t) = -\lambda - 2\sqrt{\alpha+\beta^2}\,[\cot(x+\lambda t) + \tan(x+\lambda t)]$.
$v_{24}(x,t) = -2(\alpha+\beta^2) - \frac{1}{2}\lambda^2 - \frac{1}{2}[u_{24}^2(x,t) + 2\lambda u_{24}(x,t) -$



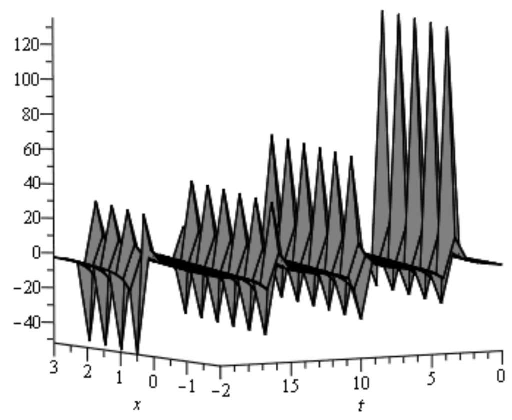


Figure 2. (*a*) Graphic of the periodic solution u_{14} (*b*) Graphic of the periodic solution u_{13}

$4\beta\sqrt{\alpha+\beta^2}(2-\cot^2(x+\lambda t)+\tan^2(x+\lambda t))]$.

For direct-viewing analysis, we provide the figures of $u_{13}(x,t)$and $u_{14}(x,t)$, where we choose $\alpha=2,\beta^2=2$ and $\lambda=2$.

Where α and β are real constants, λ is arbitrary constants in section 2.1 and 2.2.

2-3. The rational solutions to Whitham-Broer-Kaup system

(1) When $A=B=0, C\neq 0$, from Table 1, $F(\xi)=-\frac{1}{C\xi+\eta}$($\eta$ is an arbitrary constant). By *case* I, the exact solution to equation (2-7) is obtained as:

$u_{25}(x,t)=-\lambda+2C\sqrt{\alpha+\beta^2}(-\frac{1}{C(x+\lambda t)+\eta})$,

$v_{25}(x,t)=-\frac{1}{2}\lambda^2-\frac{1}{2}[u_{25}^2(x,t)+2\lambda u_{25}(x,t)+4\beta C\sqrt{\alpha+\beta^2}(\frac{C}{(C(x+\lambda t)+\eta)^2})]$.

where α and β are real constants,and λ and C are arbitrary constants.

(2) When $C=B=0$ and A is an arbitrary constant, from Table 1, $F(\xi)=A\xi$. By *case* II, the exact solution to equation (2-7) can be obtained that:

$u_{26}(x,t)=-1+2\sqrt{\alpha+\beta^2}(\frac{1}{x+\lambda t})$,

$v_{26}(x,t)=-\frac{1}{2}\lambda^2-\frac{1}{2}[u_{26}^2(x,t)+2\lambda u_{26}(x,t)-4\beta\sqrt{\alpha+\beta^2}(\frac{1}{(x+\lambda t)^2})]$.

where α and β are real constants, λ, are arbitrary constants.

2-4. The exponential solutionsto Whitham-Broer-Kaup system

(1) When $B\neq 0, C=0$ and A is anarbitrary constant, from Table 1, $F(\xi)=\frac{\exp(B\xi)-A}{B}$. By *case* II, the exact solution to equation (2-7) is found to be:

$u_{27}(x,t)=(B\sqrt{\alpha+\beta^2}-1)+2A\sqrt{\alpha+\beta^2}(\frac{\exp(B(x+\lambda t))-A}{B})$.

$v_{27}(x,t)=-\frac{1}{2}\lambda^2-\frac{1}{2}[u_{27}^2(x,t)+2\lambda u_{27}(x,t)-4\beta A\sqrt{\alpha+\beta^2}(\exp(\mathrm{B}(\mathrm{x}+\lambda\mathrm{t})))]$.

By *case* III, the exact solution to equation (2-7) can be written as:

$u_{28}(x,t)=(B\sqrt{\alpha+\beta^2}-\lambda)+2A\sqrt{\alpha+\beta^2}\left(\frac{\exp(B\xi)-A}{B}\right)^{-1}$.

$v_{28}(x,t)=-\frac{1}{2}\lambda^2-\frac{1}{2}[u_{28}^2(x,t)+2\lambda u_{28}(x,t)-4\beta A\sqrt{\alpha+\beta^2}(\frac{B^2\exp(\mathrm{B}(\mathrm{x}+\lambda\mathrm{t}))}{(\exp(\mathrm{B}(\mathrm{x}+\lambda\mathrm{t})-\mathrm{A})^2})]$.

where α and β are real constants,and λ and C are arbitrary constants.

Discussion

In this study, we aimed to present an improved F-expansion method for generating traveling wave solutions of nonlinear partial differential equation (NLPDE). The merit of the method is that it is independent of the integrability of the coupled NLPDEs, so it can be used to solve both integrable and non-integrable coupled NLPDEs. This new method is used to get some types of traveling wave solutions including the periodic waves and solitary waves for the Whitham-Broer-Kaup System. It is found that the coupled nonlinear system possesses many more solution structures. For each coupled system investigated, we are able to replicate solutions previously derived in literature and discover many new ones as well. Figures (1-2) graphically exhibit the representative structures of each explicit solution found for some special parameter values. Moreover, with the aid of computer symbolic systems (*Mathematica*or *Maple*), the method can be conveniently operated.

Altough these new solutions may be important for physical problems, this study suggests that one may find different solutions by choosing different methods. Therefore, this method can be utilized to solve many systems of nonlinear partial differential equations arising in the theory of soliton and other related areas of research. Finally, it is worthwhile to mention that the proposed method is straightforward and concise.In future studies that we plan to carry out, more applications to other nonlinear physical systems would be considered.

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