Journal of Sciences, Islamic Republic of Iran 25(1): 51- 55 (2014) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Weak*-closed invariant subspaces and ideals of semigroup algebras on foundation semigroups

B. Mohammadzadeh*

Department of Mathematics, Faculty of Sciences, Babol University of Technology, Babol, Mazandaran, Islamic Republic of Iran

Received: 27 October 2013/ Revised: 27 January 2014/ Accepted: 24 February

Abstract

Let S be a locally compact foundation semigroup with identity and $M_a(S)$ be its semigroup algebra. Let X be a weak*-closed left translation invariant subspace of $M_a(S)^*$. In this paper, we prove that X is invariantly complemented in $M_a(S)^*$ if and only if the left ideal $X_\perp$ of $M_a(S)$ has a bounded approximate identity. We also prove that a foundation semigroup with identity S is left amenable if and only if every complemented weak*-closed left translation invariant subspace of $L^\infty(S, M_a(S))$ is invariantly complemented in $L^\infty(S, M_a(S))$.

Keywords: Complemented subspace; Foundation semigroup; Semigroup algebras.

Introduction

Throughout this paper, S denotes a locally compact Hausdorff topological semi- group. By $M(S)$ we denote the space of all bounded complex regular Borel measures on S. This space with the convolution multiplication * and the total variation norm defines a Banach algebra. The space of all measures $\mu \in M(S)$ for which the maps $x \to \delta_x * |\mu|$ *and* $x \to |\mu| * \delta_x$ *from* S into $M(S)$ are weakly continuous is denoted by $M_a(S)$, where δ_x is the Dirac measure at x. The semigroup S is called foundation semigroup if it coincides with the closure of the set $\cup\{supp(\mu) : \mu \in M_a(S)\}$.

We note that $M_a(S)$ is a closed two-sided L-ideal of $M(S)$; see [1]. Let us point out that the second dual $M_a(S)^{**}$ of $M_a(S)$ is a Banach algebra with the first Arens product $\odot$ defined by

$$(F \odot H)(f) = F(Hf),$$
$$(Hf)(\mu) = H(f\mu), \text{ and } (f\mu)(\nu) = f(\mu * \nu)$$

for all $\boldsymbol{F}, \boldsymbol{H} \in \boldsymbol{M_a(S)}^{**}, \boldsymbol{f} \in \boldsymbol{M_a(S)}^*$, and $\boldsymbol{\mu}, \boldsymbol{\nu} \in \boldsymbol{M_a(S)}$.

Denote by $L^\infty(S, M_a(S))$ the set of all complex-valued bounded functions g on S that are μ-measurable for all $\mu \in M_a(S)$. We identify functions in $L^\infty(S, M_a(S))$ that agree μ-almost everywhere for all, $\mu \in M(S)$.

For every, $g \in L^\infty(S, M_a(S))$, define $\|g\|_\infty = sup\{\|g\|_{\infty,|\mu|} : \mu \in M_a(S)||\}$, where $\|.\|_{\infty,|\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. Observe that $L^\infty(S, M_a(S))$ with the complex conjugation as involution, the pointwise operations and the norm $\|.\|_\infty$ is a commutative C*-algebra.

* Corresponding author: Tel: +981113234203; Fax: +981113234203; Email: b.mohammadzadeh@nit.ac.ir
2000 Mathematics Subject Classification: 43A60, 22A20.

The duality $\tau(g)(\mu) := \mu(g) = \int_S g d\mu$ defines a linear mapping τ from $L^\infty(S, M_a(S))$ into $M_a(S)^*$. It is well-known that if S is a foundation semigroup with identity, then τ is an isometric isomorphism of $L^\infty(S, M_a(S))$ onto $M_a(S)^*$; see Proposition 3.6 of [8]. Given any $\mu \in M_a(S)$ and $g \in L^\infty(S, M_a(S))$, define the complex-valued functions $\mu \circ g$ on S by $(\mu \circ g)(x) = \mu(g_x)$ for all $x \in S, (g_x)(y) = g(yx)$ for all $y \in S$.

Bekka in [2] has proved that a weak*-closed left translation invariant subspace X of $L^\infty(G)$ is invariantly complemented if and only if $X_\perp = \{f \in L^1(G) : \langle f, \phi\rangle = 0,\ \phi \in X\}$ has a bounded approximate identity when G is a locally compact group. Also, weak*-closed translation invariant subspace in the semigroup algebra of a locally compact topological foundation semigroup to be completely complemented studied by A. Ghaffari in [4]. See, also [5]. In this work, our main purpose is to generalize Bekka's results for a foundation semigroup with identity.

Results

Let X be a weak* -closed left translation invariant subspace of $L^\infty(S, M_a(S))$. Then X is said to be invariantly complemented in $L^\infty(S, M_a(S))$ if X has a closed left translation invariant complemented in $L^\infty(S, M_a(S))$ or equivalently, if X is the range of a bounded projection on $L^\infty(S, M_a(S))$ commuting with left translations. Indeed, if X has a closed left translation invariant complement in $L^\infty(S, M_a(S))$, then there exists a closed left translation invariant subspace Y of $L^\infty(S, M_a(S))$ such that $L^\infty(S, M_a(S)) \cong X \oplus Y$. Define a mapping P on $L^\infty(S, M_a(S))$ such that $P(f_1 + f_2) = f_1$ where $f = f_1 + f_2 \in L^\infty(S, M_a(S))$. It is routine to show that P satisfies the conditions of a bounded projection commuting with left translations.

On the other hand if P is a bounded projection commuting with left translations on $L^\infty(S, M_a(S))$ with its range X, then the kernel of P is a left translation invariant subspace of complemented to X.

Also, X is called a topologically invariantly complemented in $L^\infty(S, M_a(S))$

if X is the range of a bounded projection on $L^\infty(S, M_a(S))$ such that

$$P(\mu \circ f) = \mu \circ P(f)$$

for all $\mu \in M_a(S)$ and $f \in L^\infty(S, M_a(S))$.

Let LUC(S) be the space of all left uniformly continuous functions on S; recall that a function $g \in C_b(S)$ is called left uniformly continuous if the mapping $x \to {}_x g$ from S into $C_b(S)$ is $\|.\|_\infty$-continuous, where $C_b(S)$ denotes the space of all bounded continuous complex-valued functions on S. It is not hard to see that $\mu \circ g \in \mathrm{LUC(S)}$ with $\|\mu \circ g\|_\infty \le \|g\|_\infty \|\mu\|$ for all $\mu \in M_a(S)$ and $f \in L^\infty(S, M_a(S))$. Recall that $\mathrm{LUC(S)}^*$ is a Banach algebra with the Arens product defined by

$$\langle n.m, f\rangle = \langle n, m.f\rangle \quad (n, m \in \mathrm{LUC(S)}^*, f \in LUC(S)),$$

where $(m.f)(x) = \langle m, {}_x f\rangle$ for all $\in S$.

The notion of (topological) invariantly complemented subspace of LUC(S) is defined in a similar way. We commence with the following proposition.

Proposition 2.1 Let S be a foundation semigroup with identity e. If I is a closed left ideal in $M_a(S)$, then its annihilator

$$I^\perp = \{f \in L^\infty(S, M_a(S)) : \langle f, \mu\rangle = 0 \quad \mu \in I\}$$

is a weak*- closed left translation invariant subspace of $L^\infty(S, M_a(S))$. Conversely, if X is a weak*- closed left translation invariant subspace of $L^\infty(S, M_a(S))$, then its annihilator $X_\perp$ is a closed left ideal in $\mathrm{M_a}(S)$.

Proof. Recall that $\langle \mu \circ f, \nu\rangle = \langle f, \mu * \nu\rangle$ for all $\mu, \nu \in M_a(S)$ and $f \in L^\infty(S, M_a(S))$. If I is a left ideal in $M_a(S)$ then obviously $I^\perp$ is a weak*-closed subspace of $L^\infty(S, M_a(S))$ and if (μ_α) is a bounded approximate identity of $M_a(S)$ (see [6]), then we have

$$\begin{aligned}\langle \mu, {}_x f\rangle &= \lim_\alpha \langle \mu_\alpha * \mu, {}_x f\rangle \\ &= \lim_\alpha \langle \mu, \mu_\alpha \circ {}_x f\rangle \\ &= \lim_\alpha \langle \mu, (\delta_x * \mu_\alpha) \circ f\rangle \\ &= \lim_\alpha \langle (\delta_x * \mu_\alpha) * \mu, f\rangle = 0\end{aligned}$$

for each $x \in S$ which implies that $I^\perp$ is left translation invariant.

Conversely, let X be a weak* -closed left translation invariant subspace of $L^\infty(S, M_a(S))$. Let $\nu \in X_\perp$ and $\mu \in M_a(S)$. Then it is easy to see that

$\langle \mu * \nu,\ f \rangle = 0$. It follows that $X_\perp$ is a left ideal of $M_a(S)$. Clearly X is closed. Hence $X_\perp$ is a closed left ideal of $M_a(S)$, as required.

Theorem 2.2 Let S be a foundation semigroup with identity e, X be a weak*-closed left translation invariant subspace of $L^\infty(S, M_a(S))$ and let $X_\perp$ be the annihilator of X in $M_a(S)$. Then the following assertions are equivalent:

(i) X is topologically invariantly complemented in $L^\infty(S, M_a(S))$.

(ii) X is invariantly complemented in $L^\infty(S, M_a(S))$.

(iii) $X \cap LUC(S)$ is invariantly complemented in $LUC(S)$.

(iv) $X \cap LUC(S)$ is topologically invariantly complemented in $LUC(S)$.

(v) The closed left ideal $X_\perp$ has a bounded right approximate identity.

Proof. $(i) \Longrightarrow (ii)$. Let P be a projection from $L^\infty(S, M_a(S))$ into X defined with $P(\mu \circ f) = \mu \circ P(f)$ for all $\mu \in M_a(S)$ and $f \in L^\infty(S, M_a(S))$. Let also (μ_α) be a bounded approximate identity of $M_a(S)$. By the equation $(\mu \circ f)(\nu) = f(\nu * \mu)$ we have

$$\begin{aligned}\langle \mu, P({}_x f)\rangle &= \lim_\alpha \langle \mu_\alpha * \mu, P({}_x f)\rangle \\ &= \lim_\alpha \langle \mu, \mu_\alpha \circ P({}_x f)\rangle \\ &= \lim_\alpha \langle \mu, P(\mu_\alpha \circ {}_x f)\rangle \\ &= \lim_\alpha \langle \mu, P((\delta_x * \mu_\alpha) \circ f)\rangle \\ &= \lim_\alpha \langle \mu, (\delta_x * \mu_\alpha) \circ P(f)\rangle \\ &= \lim_\alpha \langle \mu, \mu_\alpha \circ {}_x P(f)\rangle \\ &= \langle \mu, {}_x P(f)\rangle = 0\end{aligned}$$

which implies that $P({}_x f) = {}_x P(f)$ and so, X is invariantly complemented in $L^\infty(S, M_a(S))$.

To show $(ii) \Longrightarrow (iii)$; let $P: L^\infty(S, M_a(S)) \to X$ be a bounded projection commuting left translation. If $f \in LUC(S)$, then

$$\begin{aligned}\|{}_x P(f) - {}_y P(f)\|_\infty &= \|P({}_x f) - P({}_y f)\|_\infty \\ &= \|P({}_x f - {}_y f)\|_\infty \\ &\le \|P\| \|{}_x f - {}_y f\|_\infty .\end{aligned}$$

Hence $P(f) \in LUC(S)$; that is $P(f) \in X \cap LUC(S)$. On the other hand, if we have $f \in X \cap LUC(S)$, then $f \in X$ and so, $Pf = f$. By the definition of invariantly complemented of $X \cap LUC(S)$, it is an invariantly complemented subspace of $LUC(S)$, as required.

For the proof of implication $(iii) \Longrightarrow (iv)$ it is enough to show that the mapping P which is taken in the implication $(ii) \Longrightarrow (iii)$ satisfies the equality

$$P(\mu \circ f) = \mu \circ P(f)$$

for all $\mu \in M_a(S)$ and $f \in L^\infty(S, M_a(S))$. To this end, let $f \in LUC(S)$. Since $\mu \circ f$ is an element of $LUC(S)$, we have $P(\mu \circ f) = \mu \circ P(f)$.

To prove $(iv) \Longrightarrow (v)$, let P be a bounded projection from $LUC(S)$ onto $X \cap LUC(S)$ with $P(\mu \circ f) = \mu \circ P(f)$ for all $\mu \in M_a(S)$ and $f \in LUC(S)$.

Define

$$P': L^\infty(S, M_a(S)) \to L^\infty(S, M_a(S))$$

through

$$\langle P'f, \mu\rangle = \overline{P(\mu \circ f)(e)} \qquad \left(\mu \in M_a(S), f \in L^\infty(S, M_a(S))\right).$$

To see that the range of p' is X, first observe that when $f \in LUC(S)$ we have

$$\overline{P(\mu \circ f)(e)} = (\mu \circ Pf)(e) = \langle Pf, \mu\rangle;$$

that is, $P'_{|LUC(S)} = P$. Now, let (μ_α) be a bounded approximate identity of $M_a(S)$.

Then

$$\begin{aligned}\langle \mu, Pf\rangle &= \lim_\alpha \langle \mu * \mu_\alpha, Pf\rangle \\ &= \overline{\lim_\alpha ((\mu * \mu_\alpha) \circ f)(e)} \\ &= \overline{\lim_\alpha P(\mu \circ (\mu_\alpha \circ f))(e)} \\ &= \lim_\alpha \langle \mu, P(\mu_\alpha \circ f)\rangle = 0\end{aligned}$$

which implies that $P'f \in (X_\perp)^\perp = X$. Thus the range of P' is a subset of X.

If $f \in X$ and $\nu \in X_\perp$, then

$\langle \nu, \mu f\rangle = \langle \mu * \nu, f\rangle = 0 \quad (\mu \in M_a(S))$.

Consequently $\mu \circ f \in (X^\perp)^\perp = X$. Hence $\mu \circ f \in X \cap LUC(S)$ and

$$\langle \mu, P'f\rangle = \overline{P(\mu \circ f)(e)} = \overline{(\mu \circ f)(e)} = \langle \mu, f\rangle;$$

that is $P'f = f$ for all $f \in X$ and so, the range of P' is X. Therefore P' is a bounded projection onto X extending P.

Now let (μ_α) be a bounded approximate identity of $M_a(S)$ bounded by one. Set

$I = X_\perp$ and $C = 1 + \|P\|$. Let $E' \in L^\infty(S, M_a(S))$ be a weak*-closure point of (μ_α). Define a linear functional E on $L^\infty(S, M_a(S))$ by

$$\langle f, E\rangle = \langle f - P'f, E'\rangle \quad \left(f \in L^\infty(S, M_a(S))\right),$$

then

$$\|E\| \le (1 + \|P\|)\|E'\| \le C$$

and $\langle f, E\rangle = 0$ for all $f \in I^\perp$. Thus $E \in B_C(I^{**}) = \{F \in I^{**} : \|F\| \le C\}$ where I^{**} denotes the continuous bidual of I. By Alaoglu's Theorem $B_C(I)$ is weak*-dense in $B_C(I^{**})$, and so, there exists a net (ν_β) in $B_C(I)$ such that $\nu_\beta \to E$ with respect to $\sigma(I^{**}, I^*)$ topology. Since X is complemented in $L^\infty(S, M_a(S))$, it is easy to see that $\nu_\beta \to E$ with respect to weak* topology as well. We need only to show that (ν_β) is a weak right approximate identity for **I** (see [3]).

To this end, let $\mu \in I$ and $f \in L^\infty(S, M_a(S))$. Then we have

$$\begin{aligned}
\lim_\beta\langle \mu * \nu_\beta - \mu, f\rangle &= \lim_\beta\langle \mu * \nu_\beta\rangle - \langle \mu, f\rangle \\
&= \lim_\beta\langle \nu_\beta, \mu \circ f\rangle - \langle \mu, f\rangle \\
&= \langle \mu \circ f - P'(\mu \circ f), E'\rangle - \langle \mu, f\rangle \\
&= \langle \mu \circ f, E'\rangle - \langle \mu, f\rangle - \langle P'(\mu \circ f), E'\rangle \\
&= \lim_\alpha\langle \mu_\alpha, \mu \circ f\rangle - \langle \mu, f\rangle \\
&\quad - \lim_\alpha\langle \mu_\alpha, P'(\mu \circ f)\rangle \\
&= \lim_\alpha\langle \mu * \mu_\alpha, f\rangle - \langle \mu, f\rangle \\
&\quad - \lim_\alpha \overline{\left((\mu_\alpha * \mu) \circ f\right)(e)} \\
&= - \lim_\alpha \overline{\left((\mu_\alpha * \mu) \circ f\right)(e)} \\
&= -\langle \mu, P'f\rangle = 0 \ (since \ P'f \in X)
\end{aligned}$$

which implies that (ν_β) is a bounded weak right approximate identity. This completes the proof of the implication $(iv) \Longrightarrow (v)$.

Finally to show $(v) \Longrightarrow (i)$, let (ν_β) be a bounded right approximate identity for $X_\perp$ and $E \in L^\infty(S, M_a(S))$ be the weak*-limit point of the net (ν_β). Define

$$P: L^\infty(S, M_a(S)) \to L^\infty(S, M_a(S))$$

by

$$\langle \mu, Pf\rangle = \langle \mu, f\rangle - \langle \mu \circ f, E\rangle \quad (\mu \in M_a(S), f \in L^\infty(S, M_a(S))).$$

Clearly P is a bounded projection with range X. For all $\mu, \nu \in M_a(S)$ and $f \in L^\infty(S, M_a(S))$, since

$$\begin{aligned}
\langle \nu, P(\mu \circ f)\rangle &= \langle \nu, \mu \circ f\rangle - \langle \nu \circ (\mu \circ f), E\rangle \\
&= \langle \mu * \nu, f\rangle - \langle (\mu * \nu) \circ f, E\rangle \\
&= \langle \nu, \mu \circ P(f)\rangle.
\end{aligned}$$

We have then X is topologically invariantly complemented in $L^\infty(S, M_a(S))$ as required. This completes the proof of the theorem.

Before the next theorem, recall that a foundation semigroup S is called left amenable if there exists a left invariant mean on $L^\infty(S, M_a(S))$ (or equivalently, on LUC(S)).

Theorem 2.3 Let S be a foundation semigroup with identity. Then the following assertions are equivalent:

(i) S is left amenable.

(ii) Every complemented weak*-closed left translation invariant subspace of $L^\infty(S, M_a(S))$ is invariantly complemented in $L^\infty(S, M_a(S))$.

(iii) Every complemented weak*-closed left translation invariant subspace of $L^\infty(S, M_a(S))$ is topologically invariantly complemented in $L^\infty(S, M_a(S))$.

(iv) A closed left ideal I of $M_a(S)$ has a bounded right approximate identity if and only if its annihilator $I^\perp$ is complemented in $L^\infty(S, M_a(S))$.

Proof. Theorem 2.2 shows that (ii), (iii) and (iv) are equivalent. To see (ii)⟹(i), let $I_0(M_a(S)) := \{\mu \epsilon M_a(S) : \mu(S) = 0\}$. Observe that $I_0(M_a(S))^\perp = \mathbb{C}.1_S$ and so the left ideal $I_0(M_a(S))$ has a right bounded approximate identity. From [7] we conclude that S is left amenable, as required.

(i)⟹(ii). Let X be a weak*-closed left translation invariant subspace of $L^\infty(S, M_a(S))$ and let $P: L^\infty(S, M_a(S)) \to X$ be a bounded projection on to X. Let M be a left invariant mean on $L^\infty(S, M_a(S))$. For $f \in LUC(S)$ and $\mu \in M_a(S)$ the mapping

$$x \longrightarrow \langle\, _xP(\,_xf), \mu\rangle$$

is a continuous and bounded function on S. Define an operator $Q: LUC(S) \longrightarrow L^\infty(S, M_a(S))$ by

$$\langle Q(f), \mu\rangle = M\left(x \longrightarrow \langle\, _xP(\,_xf), \mu\rangle\right).$$

It is readily verified that $\|Q\| \le \|P\|$, Q commutes with left translations, $Q(f) = f$ for all $f \in X \cap$

$LUC(S)$ and $Q(LUC(S)) \subseteq X \cap LUC(S)$. Thus $X \cap LUC(S)$ is invariantly complemented in $LUC(S)$. Theorem 2.2 shows that X is invariantly complemented in $L^{\infty}(S, M_a(S))$. The proof is complete.

References

1. Baker A.C. and Baker J.W. Algebra of measures on locally compact semigroups III. *J. London Math. Soc.,* **4**: 685-695 (1972).
2. Bekka E. B. *Complemented subspace of* $L^{\infty}(G)$, *ideals of* $L^{1}(G)$ *and amenability. Monatsh. Math.* **109**: 195-203(1990).
3. Bonsall F. and Duncan J. Complete Normed Algebras. Springer-Verlag, New York (1973).
4. Ghaffari A. Projection on weak*-closed subspace of dual Banach Algebra. *Acta Math. Sinica, English series,* **28**: 477-486(2012).
5. Ghaffari A. Projection on to invariant subspace of some Banach Algebra. *Acta Math. Sinica, English series,* **24:** 1089-1096 (2008).
6. Lashkarizadeh Bami M. On the multipliers of (*Ma*(*S*),*L*∞(*S*,*Ma*(*S*))) of a foundation semigroup *S*. *Math.Nachr.*, **181**: 73-80 (1996).
7. Lau A .T . Analysis on a class of Banavh algebras with application to harmonic analysis on locally compact groups and semigroups. *Fund. Math.* **118**: 161-175(1983).
8. Sleijpen G .L. *The* dual of the space of measures with continuous translations. *Semigroup Forum* **22**: 139-150 (1981).