Journal of Sciences, Islamic Republic of Iran 25(3): 265 - 271 (2014) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Structure of Certain Banach Algebra Products

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Received: 24 May 2014 / Revised: 24 June 2014 / Accepted: 21 July 2014

Abstract

Let A and B be Banach algebras, $\alpha,\beta \in Hom(A,B)$, $\|\alpha\| \le 1$ and $\|\beta\| \le 1$. We define an (α,β)-product on $A \times B$ which is a strongly splitting extension of A by B. We show that these products form a large class of Banach algebras which contains all module extensions and triangular Banach algebras. Then we consider spectrum, Arens regularity, amenability and weak amenability of these products.

Keywords: Module extension, (α,β)-product, Arens regularity, Amenability, Weak amenability.

Introduction

Let A and B be Banach algebras and α be a multiplicative linear functional on A. The Lau product $A \times_\alpha B$ was first introduced by Lau [10] for the special case that A is the predual of a von Neumann algebra and α is the identity of A^*. (Our notation varies from that of [10, 11] due to some reasons which will be seen later). Lau used this product as a tool in the study of certain Banach algebras associated with locally compact groups and semigroups. Monfared [11] extended the notion of Lau product $A \times_\alpha B$ to arbitrary Banach algebras and studied various properties of such products. In particular $A \times_\alpha B$ is a strongly splitting Banach algebra extension of B by A. Motivated by Wedderburn's principal theorem, splitting of Banach algebra extensions has been a major question in the theory of Banach algebras; See [13, 1] for a through study of this question and its relation to automatic continuity and cohomology of Banach algebras.

Module extensions as generalizations of Banach algebra extensions were introduced by Gourdeau [8] and were used to show that amenability of A^{**} implies amenability of A. Zhang [15] used module extensions to answer an open question regarding weak amenability, raised by Dales, Ghahramani, and Gronbaek [3]. Monfared [11, page 279] has pointed out that an effort to generalize the product in the following way, involving two characters $\alpha,\beta \in \Delta(A)$,

$$(a,b)(a',b') = (aa', \alpha(a)b' + \beta(a')b + bb')$$

would lead to a non-associative product, unless $\alpha = \beta$. However dropping the term bb' in the above identity and taking α and β to be arbitrary would lead to an associative multiplication which generalizes product of module extensions. Inspired by this modification, we define (α,β)-product by the following identity, where α and β are homomorphisms from A into B.

$$(a,b)\cdot(a',b') = (aa', \alpha(a)b' + b\beta(a')).$$

As we will see in example 2.3, triangular Banach algebras can be represented in terms of an (α,β)-

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product. Besides the above mentioned group of examples, in contrast to direct products, (α,β)-products provide a wealth of counter-examples, as there are properties such as commutativity, which are satisfied by two of A, B, $A\times_{\alpha,\beta}B$, but not by the third one.

These facts suggest that (α,β)-products are worth to study. In the present paper we will consider basic algebraic properties, spectrum, Arens regularity, amenability, and weak amenability of (α,β)-products. In the forthcoming paper we will study (α,β)-amenability and (α,β)-weak amenability of arbitrary Banach algebras, with a new approach, which brings several notions of amenability under one roof. See also [4] for some related results in this direction.

Before proceeding further, let us recall some terminology.

Throughout A and B are Banach algebras, $Hom(A,B)$ denotes the set of all homomorphisms from A into B and by $\Delta(A)$ we mean $Hom(A,\mathbb{C})$. Recall that an extension of A by B is a short exact sequence

$$\Sigma: 0\to B\overset{i}{\to}U\overset{q}{\to}A\to 0$$

of Banach algebras and continuous algebra homomorphisms. The extension Σ splits strongly if there is a continuous homomorphism $\theta: A\to U$ such that $qo\theta=I_A$.

Results and Discussion

1. The Banach algebra $A\times_{\alpha,\beta}B$

In this section we study some properties of the (α,β)-product. We begin with a more general definition, namely $A\times_{\alpha,\beta}X$ where X is a Banach B-bimodule, as it was appeared in the forthcoming paper.

Definition 1.1 *Let X be a Banach B-bimodule, $\alpha,\beta\in Hom(A,B)$, $\|\alpha\|\le 1$ and $\|\beta\|\le 1$. The Banach algebra $A\times_{\alpha,\beta}X$ is defined as the l^1-direct product $A\times X$ with multiplication*

$$(a_1,x_1)\cdot(a_2,x_2)=(a_1a_2,\alpha(a_1)x_2+x_1\beta(a_2))$$
$$((a_1,x_1),(a_2,x_2)\in A\times_{\alpha,\beta}X).$$

Example 1.2 *In the above definition if we assume $A=X$ and $\alpha=\beta=id$, then $A\times_{\alpha,\beta}X$ is the module extension of A as it was defined by Gourdeau in [8].*

Example 1.3 *Suppose A and B are Banach algebras and X is a Banach (A, B)-module. The triangular algebra $T=\begin{pmatrix}A & X\\ 0 & B\end{pmatrix}$ with usual matrix operations and norm*

$$\left\|\begin{pmatrix}a & x\\ 0 & b\end{pmatrix}\right\|=\|a\|_A+\|x\|_X+\|b\|_B$$

is a Banach algebra. For more information on T see [6].

We may trun X into an $A\oplus_1 B$-bimodule ($\oplus_1$ denotes the l^1-direct sum) with module actions

$$(a,b).x=ax,\quad x.(a,b)=xb,$$
$$(a\in A, b\in B, x\in X).$$

Also we may define

$$\alpha,\beta\in Home((A\oplus_1 B),(A\oplus_1 B)),\ \alpha(a,b)=(a,0), \beta(a,b)=(0,b).$$

Then one can easily see that the map

$$\theta:(A\oplus_1 B)\times_{\alpha,\beta}X\to T, \theta((a,b),x)=\begin{pmatrix}a & x\\ 0 & b\end{pmatrix}$$

is a surjective isometric algebra isomorphism.

Remark 1.4 (i) *Let A and B be Banach algebras. $A\times_{\alpha,\beta}B$ is a strongly splitting Banach algebra extension of A by B. In other words, B is a closed ideal of $A\times_{\alpha,\beta}B$ and $(A\times_{\alpha,\beta}B)/B$ is isometrically isomorphic to A.*

(ii) $A\times_{\alpha,\beta}B$ is commutative if and only if A is commutative and $\alpha(a)b=b\beta(a)\quad(a\in A, b\in B)$.

(iii) For $\alpha,\beta,\gamma,\eta\in Hom(A,B)$, $A\times_{\alpha,\beta}B\cong A\times_{\gamma,\eta}B$ if and only if there exist $\varphi,\psi\in Hom(A)$ such that $\alpha=\gamma o\varphi$, $\beta=\eta o\psi$, if and only if there exist $\varphi,\psi \in Hom(B)$ such that $\alpha = \varphi o \gamma$, $\beta = \psi o \eta$.

(iv) The dual of the space $A \times_{\alpha,\beta} B$ can be identified with $A^* \times B^*$ naturally as in the direct products.

(v) Suppose I is an ideal of A and J is an ideal of B. Then

(a) If $I \subseteq Ker\alpha \cap Ker\beta$, then $I \times J$ is an ideal in $A \times_{\alpha,\beta} B$.

(b) If $I \not\subseteq Ker\alpha \cap Ker\beta$, then $I \times J$ is an ideal in $A \times_{\alpha,\beta} B$ if and only if $J = B$.

Example 2.3, the preceding remark and the next proposition reveal resemblance of (α,β)-products to matrix products.

Proposition 1.5 Let M be an ideal of $A \times_{\alpha,\beta} B$ and

$$I = \{a \in A : (a,b) \in M \quad \text{for some} \quad b \in B\},$$
$$J = \{b \in B : (a,b) \in M \quad \text{for some} \quad a \in A\}.$$

Then

(i) I is an ideal in A.

(ii) If α and β are onto, then J is an ideal of B. Furthermore if A has an approximate identity and M is closed, then $M = I \times J$.

Proof. **(i)** Straightforward.

(ii) Let $j \in J$ and $b \in B$. Then there are $a, a' \in A$ such that $\alpha(a) = \beta(a') = b$. Since M is an ideal of $A \times_{\alpha,\beta} B$, then $(a,b)(0,j) = (0,\alpha(a)j)$ and $(0,j)(a,b) = (0, j\beta(a'))$ are both in M and hence $jb, bj \in J$.

Let $(a_\lambda)_\lambda$ be a bounded approximate identity for A, $a_0 \in I$ and $b_0 \in J$. Choose $a \in I$ and $b \in J$ such that $(a,b_0) \in M$ and $(a_0,b) \in M$. Then

$$\|(a_\lambda,0)(a_0,0) - (a_0,0)\| = \|a_\lambda a_0 - a_0\| \to 0$$

and hence $(a_0,0) \in M$. Similarly $(a,0) \in M$. Therefore

$$(a_0,b_0) = (a_0,0) + (a,b_0) - (a,0) \in M.$$

Proof of the next theorem was inspired by [11, proposition 2.4.]

Theorem 1.6 *Let A and B be Banach algebras with the non-empty spectrum, $\alpha,\beta \in Hom(A,B)$, $\|\alpha\| \le 1$ and $\|\beta\| \le 1$.*

Let $E := \{(1/2(\psi o(\alpha+\beta),\psi) : \psi \in \Delta(B)\}$ and $F := \{(\varphi,0) : \varphi \in \Delta(A)\}$. Then E and F are disjoint, closed subsets of $\left(\Delta(A \times_{\alpha,\beta} B), weak^*\right)$ and $\Delta(A \times_{\alpha,\beta} B) = E \cup F$.

Proof. It is easy to see that $E \cup F \subseteq \Delta(A \times_{\alpha,\beta} B)$ and $E \cap F = \phi$. Conversely, let $(\varphi,\psi) \in \Delta(A \times_{\alpha,\beta} B)$. Then the identities

$$(\varphi,\psi)((a,b)(a',b')) = (\varphi,\psi)(a,b)(\varphi,\psi)(a',b'),$$
$$(a,b),(a',b') \in A \times_{\alpha,\beta} B$$

imply that

$$\varphi(aa') + \psi(\alpha(a)b' + b\beta(a')) = \varphi(a)\varphi(a') + \varphi(a)\psi(b') + \varphi(a')\psi(b) + \psi(b)\psi(b').$$

Taking $b = b' = 0$, we get $\varphi(aa') = \varphi(a)\varphi(a')$, and taking $a = a' = 0$, we get $\psi(b)\psi(b') = 0$. Thus $\psi(\alpha(a))\psi(b') + \psi(b)\psi(\beta(a')) = \varphi(a)\psi(b') + \varphi(a')\psi(b)$.

Taking $a = a'$, $b = b'$, we get $\psi(b)(\psi(\alpha(a) + \beta(a)) = 2\varphi(a)\psi(b)$. So if $\psi \ne 0$ and $b \in B$ is chosen so that $\psi(b) \ne 0$ then, $\varphi = 1/2(\psi o(\alpha+\beta)$. Therefore $(\varphi,\psi) \in E$.

Now if $\psi = 0$, then $(\varphi,0) \in F$. Therefore $\Delta(A \times_{\alpha,\beta} B) = E \cup F$.

Let $(1/2(\psi_0 o(\alpha+\beta),\psi_0) \in E$ and choose $b \in B$ such that $\psi_0(b) \ne 0$. Let $\varepsilon = 1/2|\psi_0(b)|$ and consider the following relative weak*-neighborhood of $(1/2(\psi_0 o(\alpha+\beta),\psi_0)$

$$U = \{(\varphi,\psi) \in \Delta(A \times_{\alpha,\beta} B) : |\psi(b) - \psi_0(b)| < \varepsilon\}.$$

If $(\varphi,0)\in U\cap F$, then $|\psi_0(b)|<\varepsilon$, which is a contradiction. Thus $U\subseteq E$. This shows that E is open in $\left(\Delta(A\times_{\alpha,\beta}B),weak^*\right)$ and hence F is closed.

Suppose $(\varphi,0)\in F\cap\overline{E}^{w^*}$ and choose a net $\{(1/2(\psi_\lambda o(\alpha+\beta)\psi_\lambda)\}$ in E which is weak*-convergent to $(\varphi,0)$, that is,

$$1/2(\psi_\lambda o(\alpha+\beta)(a)+\psi_\lambda(b)\to\varphi(a)$$
$$(a,b)\in A\times_{\alpha,\beta}B.$$

Taking $a=0$, we conclude that $\psi_\lambda(b)\to 0, b\in B$. In particular $\psi_\lambda o(\alpha+\beta)\xrightarrow{w^*}0$. Letting $b=0$ we see that $1/2(\psi_\lambda o(\alpha+\beta)(a)\to\varphi(a)$ and hence $\varphi=0$ which is a contradiction. Therefore E is closed in $\left(\Delta(A\times_{\alpha,\beta}B),weak^*\right)$.

Corollary 1.7 *Let* $\alpha,\beta\in Hom(A,B)$, $\|\alpha\|\le 1$ *and* $\|\beta\|\le 1$. *Then* $A\times_{\alpha,\beta}B$ *is semisimple if and only if* A *and* B *are semisimple.*

Proof. Suppose $A\times_{\alpha,\beta}B$ is semisimple, and $b\in B$ is such that for $\psi\in\Delta(B)$, $\psi(b)=0$. Then $(1/2(\psi o(\alpha+\beta),\psi)(0,b)=0$ and $(\varphi,0)(0,b)=0\ (\varphi\in\Delta(A))$. Thus $(\varphi,\psi)(0,b)=0$ for all $(\varphi,\psi)\in\Delta(A\times_{\alpha,\beta}B$ and hence $b=0$. Therefore B is semisimple. Similarly A is semisimple.

Conversely if $(a,b)\in A\times_{\alpha,\beta}B$ is so that for $(\varphi,\psi)\in\Delta(A\times_{\alpha,\beta}B)$, $(\varphi,\psi)(a,b)=0$, then $\varphi(a)=(\varphi,0)(a,b)=0\ (\varphi\in\Delta(A))$. Since A is semisimple, it follows that $a=0$. Consequently $\psi(b)=0$, $(\psi\in\Delta(B))$, and hence $b=0$ as B is semisimple. Therefore $A\times_{\alpha,\beta}B$ is semisimple.

Remark 1.8 *Suppose* A *is commutative and for every* $a\in A$ *and* $b\in B$, $\alpha(a)b=b\beta(a)$. *Since* B *is a closed ideal of* $A\times_{\alpha,\beta}B$ *and* $(A\times_{\alpha,\beta}B)/B$ *is isometrically isomorphic to* A, *it follows from [9, theorems 4.2.6 and 4.3.8] and part (iii) of 2.4 that* $A\times_{\alpha,\beta}B$ *is regular if and only if* A *and* B *are regular.*

2. Arens regularity

Let A be a Banach algebra. The first and second Arens multiplications on A^{**} that we denote by ∇ and $\lozenge$ respectively, are defined in three steps. For $a,b\in A,\phi\in A^*$ and $\Phi,\Psi\in A^{**}$, the elements $\phi.a, a.\phi, \Phi.\phi, \phi.\Phi$ of A^* and $\Psi\nabla\Phi, \Phi\lozenge\Psi$ of A^{**} are defined in the following way:

$$<\phi.a,b>=<\phi,ab> \qquad <a.\phi,b>=<\phi,ba>$$
$$<\Phi.\phi,b>=<\Phi,\phi b> \qquad <\phi.\Phi,a>=<\Phi,a.\phi>$$
$$<\Phi\nabla\Psi,\phi>=<\Phi,\Psi.\phi> \qquad <\Phi\lozenge\Psi,\phi>=<\Psi,\phi.\Phi>.$$

When we refer to A^{**} without explicit reference to any of Arens products, we mean A^{**} with the first Arens product. For fixed $\Psi\in A^{**}$ the map $\Phi\mapsto\Phi\nabla\Psi$ [resp. $\Phi\mapsto\Psi\lozenge\Phi$] is $\mathrm{weak}^*-\mathrm{weak}^*$ continuous, but the map $\Phi\mapsto\Psi\nabla\Phi$ [resp. $\Phi\mapsto\Phi\lozenge\Psi$] is not necessarily $\mathrm{weak}^*-\mathrm{weak}^*$ continuous, unless Ψ is in A. The left and right topological centers of A^{**} are defined by:

$$Z_t^{(l)}(A^{**})=\{\Phi\in A^{**}:\Phi\nabla\Psi=\Phi\lozenge\Psi,\quad \Psi\in A^{**}\},$$
$$Z_t^{(r)}(A^{**})=\{\Phi\in A^{**}:\Psi\nabla\Phi=\Psi\lozenge\Phi,\quad \Psi\in A^{**}\}.$$

We say that A is left Arens regular [resp. strongly Arens irregular] if $Z_t^{(l)}(A^{**})=A^{**}$ [resp. $Z_t^{(l)}(A^{**})=A$], right Arens regular [resp. strongly Arens irregular] if $Z_t^{(r)}(A^{**})=A^{**}$ [resp. $Z_t^{(r)}(A^{**})=A$], and Arens regular [resp. strongly Arens irregular] if it is both left and right Arens regular [resp. strongly Arens irregular].

Let $\alpha,\beta\in Hom(A,B)$. Then both of $\alpha^{**}:(A^{**},\nabla)\to(B^{**},\nabla)$ and $\alpha^{**}:(A^{**},\lozenge)\to(B^{**},\lozenge)$ are continuous homomorphisms [2, page 251]. Moreover if $\|\alpha\|\le 1$, then $\|\alpha^{**}\|\le 1$. A similar argument applies to β.

Proof of the next theorem was inspired by [11, proposition 2.12.]

Theorem 2.1 *Suppose* $\alpha,\beta \in Hom(A,B)$, $\|\alpha\| \le 1$, $\|\beta\| \le 1$, *and* B *is Arens regular.*

(i) If A^{**}, B^{**}, and $(A\times_{\alpha,\beta} B)^{**}$ are equipped with their first [resp. second] Arens products, then $(A\times_{\alpha,\beta} B)^{**}$ is isometrically algebra isomorphic to $A^{**}\times_{\alpha^{**},\beta^{**}} B^{**}$.

(ii) Let Z_t be either of left or right topological centers. Then $Z_t((A\times_{\alpha,\beta} B)^{**}) = Z_t(A^{**})\times_{\alpha^{**},\beta^{**}} B^{**}$.

In particular $A\times_{\alpha,\beta} B$ is Arens regular if and only if A is Arens regular.

Proof. (i) Throughout, we do not distinguish the two Banach spaces $(A\times B)^{**}$ and $A^{**}\times B^{**}$ as they can be identified in a natural way. Since the underlying Banach space of both of $(A\times_{\alpha,\beta} B)^{**}$ and $A^{**}\times_{\alpha^{**},\beta^{**}} B^{**}$ are $A^{**}\times B^{**}$, then it is enough to show that the identity map between these two algebras keeps the product. The first Arens product on $A^{**}\times_{\alpha^{**},\beta^{**}} B^{**}$ is identified by the equations

$$(\Phi,\Psi)(\Phi',\Psi') = (\Phi\nabla\Phi', \alpha^{**}(\Phi)\nabla\Psi' + \Psi\nabla\beta^{**}(\Phi')) \qquad (1)$$
$$(\Phi,\Psi),(\Phi',\Psi') \in A^{**}\times B^{**}.$$

We calculate the first Arens product on $(A\times_{\alpha,\beta} B)^{**}$. Let $(a,b),(a',b') \in A\times_{\alpha,\beta} B$, $(\varphi,\psi)\in A^{*}\times B^{*}$, and $(\Phi,\Psi),(\Phi',\Psi')\in A^{**}\times B^{**}$. Then:

$$<(\varphi,\psi)\cdot(a,b),(a',b')>=<(\varphi,\psi),(a,b)\cdot(a',b')>$$
$$=<(\varphi,\psi),(aa',\alpha(a)b'+b\beta(a'))>$$
$$=<\varphi, aa'>+<\psi,\alpha(a)b'+b\beta(a')>$$

$$=<\varphi\cdot a+\beta^{*}(\psi\cdot b), a'>+<\psi\cdot\alpha(a), b'>$$
$$=<(\varphi\cdot a+\beta^{*}(\psi\cdot b),\psi\cdot\alpha(a)),(a',b')>.$$

Thus

$$(\varphi,\psi)\cdot(a,b)=(\varphi\cdot a+\beta^{*}(\psi\cdot b),\psi\cdot\alpha(a)).$$

Also

$$<(\Phi,\Psi)\cdot(\varphi,\psi),(a,b)>=<(\Phi,\Psi),(\varphi,\psi)\cdot(a,b)>$$
$$=<(\Phi,\Psi),(\varphi\cdot a+\beta^{*}(\psi\cdot b),\psi\cdot\alpha(a))>$$

$$=<\Phi,\varphi\cdot a>+<\Phi\circ\beta^{*},\psi\cdot b>+<\Psi,\psi\cdot\alpha(a)>$$

$$=<\Phi\cdot\varphi,a>+<\beta^{**}(\Phi)\cdot\psi,b>+<\alpha^{*}(\Psi\cdot\psi),a>$$

$$=<(\Phi\cdot\varphi+\alpha^{*}(\Psi\cdot\psi),\beta^{**}(\Phi)\cdot\psi),(a,b)>.$$

So

$$(\Phi,\Psi)\cdot(\varphi,\psi)=(\Phi\cdot\varphi+\alpha^{*}(\Psi\cdot\psi),\beta^{**}(\Phi)\cdot\psi).$$

Now

$$<(\Phi,\Psi)\mathrm{W}(\Phi',\Psi'),(\varphi,\psi)>=<(\Phi,\Psi),$$
$$(\Phi',\Psi')\cdot(\varphi,\psi)>$$
$$=<(\Phi,\Psi),(\Phi'\cdot\varphi+\alpha^{*}(\Psi'\cdot\psi),\beta^{**}(\Phi')\cdot\psi)>$$

$$=<\Phi,\Phi'\cdot\varphi+\alpha^{*}(\Psi'\cdot\psi)>+<\Psi,\beta^{**}(\Phi')\cdot\psi>$$

$$=<\Phi,\Phi'\cdot\varphi>+<\alpha^{**}(\Phi),\Psi'\cdot\psi>+<\Psi,\beta^{**}(\Phi')\cdot\psi>$$
$$=<(\Phi\nabla\Phi',\alpha^{**}(\Phi)\nabla\Psi'+\Psi\nabla\beta^{**}(\Phi')),(\varphi,\psi)>.$$

Therefore

$$(\Phi,\Psi)(\Phi',\Psi') = (\Phi\nabla\Phi',\alpha^{**}(\Phi)\nabla\Psi' + \Psi\nabla\beta^{**}(\Phi')). \qquad (2)$$

The result for the first Arens product follows from (1) and (2). A similar argument provides the result for the second Arens product.

(ii) Since B is Arens regular, then $B^{**}=Z_t^{(l)}(B^{**})=Z_t^{(r)}(B^{**})$. Let

$$(\Phi,\Psi)\in Z_t^{(l)}((A\times_{\alpha,\beta} B)^{**}) = Z_t^{(l)}(A^{**}\times_{\alpha^{**},\beta^{**}} B^{**}).$$

Then for every $(\Phi',\Psi')\in A^{**}\times_{\alpha^{**},\beta^{**}} B^{**}$ we have

$$(\Phi,\Psi)\nabla(\Phi',\Psi') = (\Phi,\Psi)\lozenge(\Phi',\Psi')$$

or equivalently

$$(\Phi\nabla\Phi',\alpha^{**}(\Phi)\nabla\Psi'+\Psi\nabla\beta^{**}(\Phi')) = (\Phi\lozenge\Phi',$$
$$\alpha^{**}(\Phi)\lozenge\Psi'+\Psi\lozenge\beta^{**}(\Phi')).$$

In particular $\Phi\nabla\Phi' = \Phi\lozenge\Phi'$ and hence $\Phi\in Z_t^{(l)}$.

So

$$Z_t^{(l)}(A^{**}\times_{\alpha^{**},\beta^{**}} B^{**}) \subseteq Z_t^{(l)}(A^{**})\times_{\alpha^{**},\beta^{**}} B^{**}.$$

Conversely let $(\Phi,\Psi)\in Z_t^{(l)}(A^{**})\times_{\alpha^{**},\beta^{**}} B^{**}$. Arens regularity of B implies that

$$(\Phi,\Psi)\nabla(\Phi',\Psi')=(\Phi,\Psi)\Diamond(\Phi',\Psi')((\Phi',\Psi')\in A^{**}\times_{\alpha^{**},\beta^{**}} B^{**})$$

and hence $(\Phi,\Psi)\in Z_t^{(l)}(A^{**}\times_{\alpha^{**},\beta^{**}} B^{**})$.

Therefore

$$Z_t^{(l)}(A^{**})\times_{\alpha^{**},\beta^{**}} B^{**}\subseteq Z_t^{(l)}(A^{**}\times_{\alpha^{**},\beta^{**}} B^{**}).$$

3. Amenability

In this section we show stability of several notions of amenability with respect to the product $\times_{\alpha,\beta}$. Let X be a Banach A-bimodule. We denote the set of all bounded derivations from A into X by $Z^1(A,X)$ and the set of inner derivations from A into X by $B^1(A,X)$. Let

$$H^1(A,X):=Z^1(A,X)/B^1(A,X)$$

be the first cohomology group of A with coefficients in X. We say that A is amenable if $H^1(A,X^*)=\{0\}$ for every Banach A-bimodule X and it is weakly amenable if $H^1(A,A^*)=\{0\}$.

For a comprehensive account on amenability and weak amenability the reader is referred to the books [2, 14].

A derivation $D:A\to X$ is approximately inner if there exists a net $(x_\lambda)\subseteq X$ such that $D(a)=lim_\lambda(a\cdot x_\lambda-x_\lambda\cdot a)(a\in A)$. The algebra A is approximately amenable if for each Banach A-bimodule X every derivation $D:A\to X^*$ is approximately inner and A is approximately weakly amenable if every derivation $D:A\to A^*$ is approximately inner.

Theorem 3.1 *Let* $\alpha,\beta\in Hom(A,B)$, $\|\alpha\|\le 1$ *and* $\|\beta\|\le 1$. *Then*

(i) $A\times_{\alpha,\beta} B$ is amenable [resp. contractible] if and only if both A and B are amenable [resp. contractible].

(ii) If moreover B has a bounded approximate identity and $A\times_{\alpha,\beta} B$ is approximately amenable then so are A and B.

Proof. (i) This part follows from the fact that the short exact sequence

$$\Sigma:0\to B\overset{i}{\to}A\times_{\alpha,\beta} B\overset{q}{\to}A\to 0$$

splits strongly [1].

(ii) This is a consequence of Remark 2.4 and [7, Corollary 2.1].

The next theorem is one of the results which show the assymetry of the product $A\times_{\alpha,\beta} B$.

Theorem 3.2 *Let* $\alpha,\beta\in Hom(A,B)$, $\|\alpha\|\le 1$ *and* $\|\beta\|\le 1$.

(i) If A and B are weakly amenable then so is $A\times_{\alpha,\beta} B$.

(ii) If $A\times_{\alpha,\beta} B$ is weakly amenable then A is weakly amenable.

Moreover suppose that B is commutative.

(iii) If A and B are approximately weakly amenable then so is $A\times_{\alpha,\beta} B$.

(iv) If $A\times_{\alpha,\beta} B$ is approximately weakly amenable then A is approximately weakly amenable.

Proof. (i) Since B is a weakly amenable closed ideal of $A\times_{\alpha,\beta} B$ and $A\cong A\times_{\alpha,\beta} B/B$ is weakly amenable then $A\times_{\alpha,\beta} B$ is weakly amenable.

(ii) Let $d:A\to A^*$ be a bounded derivation and define $D:A\times_{\alpha,\beta} B\to A^*\times B^*$ by $D(a,b)=(d(a),0)$. Then D is a bounded linear map and

$$D((a,b)(a',b'))=D(aa',\alpha(a)b'+b\beta(a'))$$

$$=(d(aa'),0)=(d(a)a'+ad(a'),0)$$

$$=(d(a),0)(a',b')+(a,b)(d(a'),0)$$

$$=D(a,b)(a',b')+(a,b)D(a',b').$$

So D is a bounded derivation and hence there is a $(\zeta_1,\zeta_2)\in A^*\times B^*$ such that

$$D(a,b)=(\zeta_1,\zeta_2)(a,b)-(a,b)(\zeta_1,\zeta_2),$$
$$((a,b)\in A\times_{\alpha,\beta} B).$$

So
$(d(a),0) = D(a,0) = (\zeta_1, \zeta_2)(a,0) - (a,0)(\zeta_1, \zeta_2) = (\zeta_1 a - a\zeta_1, 0).$

Therefore, $d(a) = \zeta_1 a - a\zeta_1 \quad (a \in A)$.

(iii) Since for commutative Banach algebras the two concepts of weak amenability and approximate weak amenability coincide, then B is weakly amenable. But B is a closed ideal of $A \times_{\alpha,\beta} B$, and $A \cong A \times_{\alpha,\beta} B/B$ is approximately weakly amenable. So by [5, Proposition 2.2] $A \times_{\alpha,\beta} B$ is approximately weakly amenable.

(iv) Let $d : A \rightarrow A^*$ be a bounded derivation and as in part (ii) define a bounded derivation $D : A \times_{\alpha,\beta} B \rightarrow A^* \times B^*$ by $D(a,b) = (d(a),0)$. By assumption there exists a net $(\varphi_\lambda, \psi_\lambda)_\lambda$ in $A^* \times B^*$ such that

$$D(a,b) = lim_\lambda((a,b)(\varphi_\lambda, \psi_\lambda) - (\varphi_\lambda, \psi_\lambda)(a,b)) \quad (a,b) \in A \times_{\alpha,\beta} B.$$

Now

$$< d(a), a' > = < D(a,0), (a',0) >$$

$$= < lim_\lambda((a,0)(\varphi_\lambda, \psi_\lambda) - (\varphi_\lambda, \psi_\lambda)(a,0)), (a',0) >$$

$$= < lim_\lambda(a\varphi_\lambda - \varphi_\lambda a), a' >$$

and hence

$$d(a) = lim_\lambda(a\varphi_\lambda - \varphi_\lambda a) \quad a \in A.$$

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