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Cesaro Supermodular Order and Archimedean Copulas

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Abstract

In this paper, we introduce a new kind of order, Cesaro supermodular order, which includes supermodular order and stochastic order. For this new order, we show that it almost fulfils all desirable properties of a multivariate positive dependence order that have been proposed by Joe (1997). Also, we obtain some relations between it with other orders. Finally, we consider different issues related extended Archimedean copula and positive dependence.

Keywords: Stochastic order; Supermodular order; Positive dependent random variables; Cesaro supermodular order; Archimedean copula.

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Introduction

In recent years, dependent random variables and their concepts have been interested. Some important references are Joe (2001) and Mari and Kotz (2001) and their references among others. Several studies investigating the stochastic and supermodular orders that have the properties of positive dependence orders, have been carried out (see, for example, Muller and Scarsini (2000) and Shaked and Shanthikumar (1997)). It is known that supermodular order is strictly stronger than concordance order. Moreover, they are equivalent in the bivariate case. By comparing the sum of expectations of functions of random vectors we'll obtain a necessary and sufficient condition for new orders that we will introduce in section 2. Muller and Scarsini-(2005) characterized Archimedean copula that possesses some positive dependence properties. such as multivariate total positivity of order 2 (MTP2) and conditionally increasingness in sequence. In section 4, such characterizations will provide for an extension of Archimedean copula.

Materials and Methods

We say the function f is increasing if $x \le y$ implies $f(x) \le f(y)$.

Definition 1. A function $f: R^n \to R$ is called supermodular (or L-superaddaitive in Marshall and Olkin (١٩٧٩)) if

$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \ge f(\mathbf{x}) + f(\mathbf{y})$$

where the lattice operators and $\vee$ are defined as

$$\mathbf{x} \wedge \mathbf{y} = (\max(x_1, y_1), ..., \max(x_n, y_n)) \qquad \mathbf{x} \vee \mathbf{y} = (\min(x_1, y_1), ..., \min(x_n, y_n))$$

The following properties of supermodular functions are well-known.

Theorem 1: (a) If f is a twice differentiable function, then f is supermodular if and only if

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} \ge 0 \qquad \forall \mathbf{x} \in R^n, 1 \le i < j \le n.$$

(b) If $g_1, \ldots, g_d : R \to R$ are increasing functions and f is supermodular, then is also supermodular.

A proof of this theorem as well as many examples can be found in Marshall and Olkin (1979,P.146 ff.).

Definition 2: A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be smaller than the random vector $\mathbf{Y} = (Y, \ldots, Y_n)$ in Cesaro supermodular order k, written $\mathbf{X} \circ_{csmk} \mathbf{Y}$ if

i)

$$\sum_{i=1}^{n} E(f(X_i)) \le \sum_{i=1}^{n} E(f(Y_i)) \qquad k = 1$$

ii) $$\sum_{I_k} E(f_{I_k}(X_{i_1}, \ldots, X_{i_k}) \le \sum_{I_k} E(f_{I_k}(Y_{i_1}, \ldots, Y_{i_k}) \qquad k > 1$$

for all convex functions f and supermodular functions f_{I_k}, I_k is a subset of order k of permutation of $\{1, \ldots, n\}$, such that the expectations exist. Convex and supermodular orders are special cases, respectively for $n = 1$ and $k = n$. If f_{I_k} are increasing functions, the definition of Cesaro increasing supermodular will be yield.

For a random vector $\mathbf{X} = (X_1, \ldots, X_n)$, we denote by

$$F_{\mathbf{X}}(t) = P(X_1 \le t_1, \ldots, X_n \le t_n), \qquad t = (t_1, \ldots, t_n) \in R^n$$

the multivariate distribution function, the $F_{i_1,\ldots,i_k}$ marginal distribution function is given by

$$F_{i_1,\ldots,i_k}(t) = P(X_{i_1} \le t_{i_1}, \ldots, X_{i_k} \le t_{i_k}) \qquad t = (t_{i_1}, \ldots, t_{i_k}) \in R^k,\ 1 \le k \le n$$

and the $\bar{F}_{\mathbf{X}}$ multivariate survival function is given by

$$\bar{F}_{\mathbf{X}}(t) = P(X_1 > t_1, \ldots, X_n > t_n), \qquad t = (t_1, \ldots, t_n) \in R^n$$

Definition 3: (a) A random vector $\mathbf{X} = (X_1, \ldots, X_n)$, is said to be smaller than the random vector $\mathbf{Y} = (Y, \ldots, Y_n)$ in the upper orthant order, written $\mathbf{X} \circ_{uo} \mathbf{Y}$ (*or* $F_X \circ_{uo} F_Y$) if $\bar{F}_{\mathbf{X}}(t) \le \bar{F}_{\mathbf{Y}}(t)$ for all $t \in R^n$.

(b) A random vector $\mathbf{X} = (X_1, \ldots, X_n)$, is said to be smaller than the random vector $\mathbf{Y} = (Y, \ldots, Y_n)$ in the lower orthant order ,written $\mathbf{X} \circ_{lo} \mathbf{Y}$ (*or* $F_{\mathbf{X}} \circ_{lo} F_{\mathbf{Y}}$) if $F_{\mathbf{X}}(t) \le F_{\mathbf{Y}}(t)$ for all $t \in R^n$.

(c) A random vector $\mathbf{X} = (X_1, \ldots, X_n)$, is said to be smaller than the random vector $\mathbf{Y} = (Y, \ldots, Y_n)$ in the concordance order, written $\mathbf{X} \circ_{c} \mathbf{Y}$ (*or* $F_{\mathbf{X}} \circ_{c} F_{\mathbf{Y}}$) if both $\mathbf{X} \circ_{lo} \mathbf{Y}$ and $\mathbf{X} \circ_{uo} \mathbf{Y}$ hold.

Results and Discussion

Kimeldorf and Sampson (1989) proposed the following nine desirable properties that an ordering of distributions should have in order that higher in the ordering means more multivariate positive dependence.

Let F $(F_1, \ldots, F_n)$ be the set of random vectors, which have the same univariate marginals $F_1, \ldots, F_n$. Then a binary relation $\circ$ is said to be a multivariate positive dependence order (MPDO), if it fulfils the following properties:

(P1) (bivariate concordance) $F \circ G$ implies that $F_{i,j}(\mathbf{X}) \le G_{i,j}(\mathbf{X})$ for all $1 \le i < j \le n$ and $\mathbf{x} \in R^2$;

(P2)(reflexivity) $F \circ F$;

(P3) (transitivity) $F \circ G$ and $G \circ H$ imply $F \circ H$;

(P4) (antisymmetry) $F \circ G$ and $G \circ H$ imply $F = G$;

(P5) (bound) $F \circ F^+$ where $F^+(\mathbf{x}) = \min_i (F_i(x_i))$ is the upper Frechet bound;

(P6) (Closure with respect to convergence in distribution) $F_n \xrightarrow{L} F$ and $G_n \xrightarrow{L} G$ then $F_n \circ G_n$ for all n implies $F \circ G$;

(P7) (exchangeability) $(X_1,\ldots,X_n) \circ (Y_1,\ldots,Y_n)$ implies $\left(X_{i_1},\ldots,X_{i_n}\right) \circ \left(Y_{i_1},\ldots,Y_{i_n}\right)$ for all permutations $(i_1,\ldots,i_n)$ of $(1,\ldots,n)$;

(P8) (Closure under marginalization) $(X_1,\ldots,X_n) \circ (Y_1,\ldots,Y_n)$ implies $\left(X_{i_1},\ldots,X_{i_k}\right) \circ \left(Y_{i_1},\ldots,Y_{i_k}\right)$ for all $(i_1,\ldots,i_k)$, $2 \le k < d$;

(P9) (Invariance to increasing transforms $(X_1,\ldots,X_n) \circ (Y_1,\ldots,Y_n)$ implies $(g(X_1),\ldots,X_n) \circ (g(Y_1),\ldots,Y_n)$ for all strictly increasing functions $g: R \to R.$

We will need the following theorem. We refer to Tchen (1980) for background and details.

Theorem 2. Let $F_1,\ldots,F_n$ be n probability measures on R and let $\bar{H} = \min_i F_i$. Then for any continuous supermodular function $\varphi: R^n \to R$, $\int \varphi d\bar{H} = \sup_{H \in \mathrm{F}(F_1,\ldots,F_n)} \int \varphi dH$, if one of the following conditions holds:

(a) $\varphi \le h$ for some continuous function h such that $\int hdH$ is finite and constant for all $H \in \mathrm{F}\left(F_1,\ldots,F_n\right)$,

(b) $\{\varphi(\mathbf{X}), \mathbf{X}$ distributed as $H \in \mathrm{F}\left(F_1,\ldots,F_n\right)\}$ is uniformly integrable.

We will show now, that the Cesaro supermodular order k fulfils the multivariate positive dependence orders properties.

Theorem 3. The Cesaro supermodular order k fulfils all the properties P(1)-P(9) (with partial antisymmetry property).

Proof. The proof follows as in Muller and Scarsini(2000). Let $x, y \in R$, $1 \le i < j \le n$ and

$$f_{I_{k_0}}\left(X_{i_1},\ldots,X_{i_k}\right) = I_{(X_i > x)} I_{(X_j > y)}, \qquad i_1 = i,\, i_2 = j,\, f_{I_k} = 0,\, I_k \ne I_{k_0}.$$

$f_{I_{k_0}}$ is supermodular function. Therefore

$$\bar{F}_{i,j}(x,y) = P\left(X_i > x, X_j > y\right) = \sum\nolimits_{I_k} E(f_{I_k}\left(X_{i_1},\ldots,X_{i_k}\right))$$

$$\le \sum\nolimits_{I_k} E(f_{I_k}\left(Y_{i_1},\ldots,Y_{i_k}\right)) = \bar{G}_{ij}(x,y) = P\left(Y_i > x, Y_j > y\right)$$

Similarly, it can be shown that for each $i, j, F_{ij} \le G_{ij}$. It means that $\mathbf{X} \circ_{csmk} \mathbf{Y}$ has property (P1). (P2) and (P3) are trivial. Suppose F_{I_k} is the distribution function of $\left(X_{i_1},\ldots,X_{i_k}\right)$ and G_{I_k} is the distribution function of $\left(Y_{i_1},\ldots,Y_{i_k}\right)$. Property (P4) follows from the fact that $\mathbf{X} \circ_{csmk} \mathbf{Y}$ implies $F_{I_k} \le G_{I_k}$ as well as $\bar{F}_{I_k}(t) \le \bar{G}_{I_k}(t)$. Property (P5) follows from theorem 2 and the fact that a supermodular function of two (resp. k) variables remains supermodular .Property (P7) is obvious from the definition of supermodularity and (P8) follows immediately from Theorem 1.In Theorem 4 we will show property (P6).

Theorem 4. The following statements are equivalent:

(i) $\mathbf{X} \circ_{csmk} \mathbf{Y}$.

(ii) $\sum E(f_{I_k}(\mathbf{X})) \le \sum E(f_{I_k}(\mathbf{Y}))$ for all

bounded continuous increasing supermodular functions f_{I_k}.

(X_1, X_2)	**0**	**1**	
0	0.4	0.1	0.5
1	0.1	0.4	0.5
	0.5	0.5	

(Y_1, Y_2)	**0**	**1**	
0	0.25	0.25	0.5
1	0.25	0.25	0.5
	0.5	0.5	

Proof. It is clear(by Definition 2) that (i) implies (ii). Therefore it is sufficient to show that (ii) implies (i). One now proceeds by following the argument used in Muller and Scarsini(2000) (proof of Th. 3.3). Theorem will be proof in three step. In step 1 it will be proof for bounded semi continuous increasing supermodular functions and then for bounded functions and finally for unbounded increasing supermodular functions

Theorem 5. The stochastic order relations $\circ_{csmk}$ and $\circ_{icsmk}$ are closed with respect to weak convergence

Proof. Due to Theorem 4.2 in Muller (1997) an integral stochastic order relation is closed with respect to weak convergence, if and only if it is generated by a class of bounded continuous functions. Therefore for $\circ_{icsmk}$ the assertion follows immediately from Theorem 4. Combining This with Theorem 3.4 of Muller and Scarsini (2000) that are valid for $\circ_{csmk}$ and $\circ_{icsmk}$ yields the results for $\circ_{csmk}$.

We showed that $\circ_{csmk}$ order has the partial antisymmetry property i.e.

$$\mathbf{X} \circ_{csmk} \mathbf{Y} \qquad\qquad \mathbf{Y} \circ_{csmk} \mathbf{X}$$

imply that marginal distributions to order k are the same. But it doesn't mean that $\mathbf{X} = \mathbf{Y}$

Example 1. Let $\mathbf{X}$ and $\mathbf{Y}$ have the following distributions

$\mathbf{X} \circ_{csm1} \mathbf{Y}$ and $\mathbf{Y} \circ_{csm1} \mathbf{X}$ and but $\mathbf{X} \neq \mathbf{Y}$

Now we extend theorem 2.2 of Shaked and Shanthiku mar (1997) to Cesaro supermodular order k.

Theorem 6. Let $(X_1,\ldots,X_n)$ and $(Y_1,\ldots,Y_n)$ be two random vectors.

(a). If $(X_1,\ldots,X_n) \circ_{csmk} (Y_1,\ldots,Y_n)$ then $(g_1(X_1),\ldots,g_n(X_n)) \circ_{csmk} (g_1(Y_1),\ldots,g_n(Y_n))$ whenever $g_i : R \to R, i = 1,2,\ldots,n$ are all increasing or are all decreasing

(b). Let $\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_n$ be a set of independent random vectors where the dimension of $\mathbf{X}_i$ is $k_i, i = 1,\ldots,n$ .Let. $\mathbf{Y}_1,\mathbf{Y}_2\ldots,\mathbf{Y}_n$ be another set of independent random vectors where the dimension of $\mathbf{Y}_i$ is $k_i, i = 1,\ldots,n$. If $\mathbf{X}_i \circ_{csmk} \mathbf{Y}_i$, for $i = 1,\ldots,n$ then

$$(\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_n) \circ_{csmk} (\mathbf{Y}_1,\ldots,\mathbf{Y}_n)$$

(c). Let $\mathbf{X} = (X_1,\ldots,X_n)$ and $\mathbf{Y} = (Y_1,\ldots,Y_n)$ be two random vectors. If $\mathbf{X} \circ_{csmk} \mathbf{Y}$ then $\mathbf{X}_I \circ_{csmk} \mathbf{Y}_I$ for each $I \subseteq \{1,2,\ldots,n\}$. That is Cesaro supermodular stochastic order k is closed under marginalization.

d) Let $\mathbf{X}$ and $\mathbf{Y}$ and $\boldsymbol{\theta}$ be random vectors such that $[\mathbf{X} \mid \boldsymbol{\theta} = \theta] \circ_{csmk} [\mathrm{Y} \mid \boldsymbol{\theta} = \theta]$ for all θ in the support of $\boldsymbol{\theta}$. Then $\mathbf{X}_{\mathrm{I}} \circ_{csmk} \mathbf{Y}_{\mathrm{I}}$ .That is ,Cesaro supermodular order k is closed under mixtures.

Proof. One now proceeds by following the argument used in Shaked and Shanthikumar(1997). Part (a) follows from part (b) of Theorem 1.

In order to see part (b) let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two independent random vectors and let $\mathbf{Y}_1$ and $\mathbf{Y}_2$ be two other independent random vectors. Suppose $\mathbf{X}_1 \circ_{csmk} \mathbf{Y}_1$ and $\mathbf{X}_2 \circ_{csmk} \mathbf{Y}_2$. Then, for any family of supermodular functions $\{\varphi_{I_k}\}$ we have that

$$\sum(\mathrm{E}\left(\varphi_{I_k}(\mathbf{X}_1,\mathbf{X}_2)\right) = \mathrm{E}\sum(\mathrm{E}((\varphi_{I_k}((\mathbf{X}_1,\mathbf{X}_2)|\mathbf{X}_2) \le \sum\mathrm{E}(\mathrm{E}((\varphi_{I_k}((\mathbf{Y}_1,\mathbf{X}_2)|\mathbf{X}_2)$$

$$= \sum\mathrm{E}(\varphi_{I_k}(\mathbf{Y}_1,\mathbf{X}_2) \le \sum\mathrm{E}(\varphi_{I_k}(\mathbf{Y}_1,\mathbf{Y}_2)$$

Part (b) follows from the above by induction. Parts (c) and (d) are easy to prove.

Theorem 7. Let $\mathbf{X} \sim \mathrm{N}(\mu,\Sigma)$ and $\mathbf{Y} \sim \mathrm{N}(\mu',\Sigma')$. Then the following conditions are equivalent .

a) $\mathbf{X} \circ_{csm2} \mathbf{Y}$.

b) $\mathbf{X}$ and $\mathbf{Y}$ have the same marginals and $\sigma_{.j} \le \sigma'_{.j}$ for all i, j.

c) $\mathbf{X} \circ_{sm} \mathbf{Y}$.

Proof. The implication (a) $\Longrightarrow$ (b) follows immediately from Slepian's inequality. The implication (b) $\Rightarrow$ (c) follows from Theorem 4.2 of Muller and Scarsini (2000) . (c) $\Longrightarrow$(a) is obvious.

The notion of copula has been introduced by Sklar (1959) and studied by many authors. We refer for background, details and references to Nelsen(2006), Genesta, et. al.(2009), McNeil and Nešlehová(2009), Fermanian and Wegkamp (2012) and Arbenz(2013). Formally, given a distribution function F with marginals $F_1,\ldots,F_d$ there exists a function $C:[0,1]^d \to [0,1]$ such that, for all

$$\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathrm{R}^{\mathrm{d}}$$

$$F(\mathbf{x}) = C\left(F_1(x_1),\ldots,F_d(x_d)\right)$$

Any such function is called copula of F. Most of the multivariate-dependence structure properties of F are in the copula, which does not depend on the marginals, and is often easier to handle than the original F. More details about copula can be found in Joe(2001). A copula C_ψ is called Archimedean if it has the form

$$C_\psi = \psi\left(\sum_{i=1}^{n} \psi^{-1}(x_i)\right)$$

A trivariate generalization of Archimedean copula is

$$C_{\varphi,\psi} = \psi\left(\psi^{-1} o \varphi\left(\varphi^{-1}(x_1) + \varphi^{-1}(x_2)\right) + \psi^{-1}(x_3)\right)$$

where ψ and φ are Laplace transforms and $h = \psi^{-1} o \varphi$ is non-negative function such $(-1)^j h^{(j)} \ge 0,\ j \ge 1$ (Joe (2001),P87). If $\psi = \varphi$, it is a copula and we call it extended Archimedean copula. In this section we recall now some well-known concepts of positive dependence.

Definition 4. A copula is

(a) conditionally increasing in sequence (CIS) if $\mathbf{X}$~C and $P(X_i > t \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1})$ is increasing in $x_1, \ldots, x_{i-1}$ for all t,

(b) conditionally increasing (CI) if $\mathbf{X}$~C and $P(X_i > t \mid X_j = x_j, j \in J)$, $J \subseteq \{1, 2, \ldots, d\}, i \notin J$ is increasing in $x_j, j \in J$ and for all x_i.

(c) multivariate totally positive of order 2 (MTP2) if C has a density which is log-supermodular, i.e. if

$$\log \frac{\partial^d}{\partial x_1, \ldots, \partial x_d} C(x_1, \ldots, x_d)$$

is supermodular.

Theorem 8. Suppose $\mathbf{X} = (X_1, X_2, X_3)$ have a trivariate extended Archimedean copula $C_{\varphi,\psi}$. Then for the statements

(a) The bivariate marginal distribution (X_1, X_2), (X_2, X_3) and (X_1, X_3) is CIS,

(b) $(-\varphi)$ and $(-\psi)$ are log-convex,

(c) $P(X_3 \ge x \mid X_1 = x_1, X_2 = x_2)$ is decreasing in x_1, x_2 for all $x \in [0,1]$,

(d) $\psi^2(.)$ is log-convex ,

(e) $\mathbf{X}$ is CIS,

we have (a) ⇔ (b), (c) ⇒ (d) and(b) and (d) ⇒ (e).

Proof. From the fact that $C_{\varphi,\psi}$. is a trivariate extended Archimedean copula, follows that (X_1, X_2) ((X_2, X_3) or (X_1, X_3))is an Archimedean copula with generator φ (ψ or ψ). Then, by Theorem (2.8) of Muller and Scarsini. (۲۰۰۵) The bivariate marginal distributions are CIS iff $(-\varphi)$ and $(-\psi)$ are log-convex. Then we have (a) ⇔ (b).

$P(X_3 \le x \mid X_1 = x_1, X_2 = x_2)$ is decreasing in x_1 and x_2 for $x \in [0,1]$ iff

$$P(X_3 \le x \mid X_1 = x_1, X_2 = x_2) = \frac{\frac{\partial^2}{\partial x_1 \partial x_2} P(X_3 \le x, X_1 \le x_1, X_2 \le x_2)}{\frac{\partial^2}{\partial x_1 \partial x_2} P(X_1 \le x_1, X_2 \le x_2)}$$

$$= \frac{\frac{\partial^2}{\partial x_1 \partial x_2} \psi(\psi^{-1} o \varphi\left(\varphi^{-1}(x_1) + \varphi^{-1}(x_2)\right) + \psi^{-1}(x_3))}{\frac{\partial^2}{\partial x_1 \partial x_2} \psi(\psi^{-1} o \varphi\left(\varphi^{-1}(x_1) + \varphi^{-1}(x_2)\right))}$$

is decreasing in x_1 and x_2. Since ψ (and φ) is decreasing (and therefore ψ^{-1} (and φ^{-1}) is decreasing), then the above expression is decreasing in x_1 and x_2 iff it is increasing in $y = \psi^{-1} o \varphi$. So we want

$$\frac{\psi^{(2)}(y+z)}{\psi^{(2)}(y)}$$

to be increasing iny for all $z \in (0,\infty)$. This holds iff

$$\log\psi^{(2)}(y+z) - \log\psi^{(2)}(y)$$

is increasing in y for all $z \in (0,\infty)$; namely, $\log\psi^{(2)}$ is convex.

(b) and (d)⟹ (e) follows from Theorem (2.8) of Muller and Scarsini. (2005) and the definition of CIS.

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