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A Characterization of the Suzuki Groups by Order and the Largest Elements Order

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Abstract

One of the important problems in group theory is characterization of a group by a given property, that is, to prove there exist only one group with a given property. Let G be a finite group. We denote by $k(G)$ the largest order of elements of G. In this paper, we prove that some Suzuki groups are characterizable by order and the largest order of elements. In fact, we prove that if G is a group with $|G| = |Sz(q)|$ and $k(G) = k(Sz(q))$ where $q - 1$ or $q \pm \sqrt{2q} + 1$ is a prime number, then $G \cong Sz(q)$.

Keywords: Largest elements order; Prime graph; Frobenius Group; Suzuki group.

Introduction

For a finite group G, the set of prime divisors of $|G|$ is denoted by $\pi(G)$ and the set of order of elements of G is denoted by $\omega(G)$. Also, the largest order of elements of G is denoted by $k(G)$. Moreover, a Sylow p-subgroup of G is denoted by G_p. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two distinct vertices p and q are adjacent if and only if $pq \in \omega(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i for $i = 1,2,\dots,t(G)$. In the case where of $|G|$ is even, we always assume that $2 \in \pi_1$.

One of the important problems in group theory is characterization by given property, that is, there exist only one group with given properties (up to isomorphism). There are different kinds of characterization, for example, the characterization by the set of elements order, prime graph, the set of the number of elements with the same order. To see results on characterizing simple groups, we refer the reader to the references [11,12,14]. Recently, He and Chen studied the characterization of groups by the largest order of elements. In [7], they proved that the groups $L_2(q)$ with $q < 125$ are characterizable by their order and the largest, the second largest and the third largest order of elements. In the following, it is proved that the simple K_3-groups [6], Sporadic simple groups [1], $PGL(2,q)$, $L_2(q)$, $L_3(q)$ and $U_3(q)$, for some q [8,9,10] are characterizable by their order and the largest and the second largest and the third largest order of elements. In this article, we prove that the Suzuki groups are characterizable by their order and the largest order of elements.

1. Preliminaries

Lemma 1.1. [2,5] Let G be a Frobenius group of even order with kernel K and complement H. Then

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(a) $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;
(b) $|H|$ divides $|K| - 1$;
(c) K is nilpotent.

Definition 1.2. A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernel K/H and H respectively.

Lemma 1.3. [2] Let G be a 2-Frobenius group of even order. Then
(a) $t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;
(b) G/K and K/H are cyclic groups and $|G/K|$ divides $|\mathrm{Aut}(K/H)|$.

Lemma 1.4. [15] Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:
(a) G is a Frobenius group;
(b) G is a 2-Frobenius group;
(c) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathrm{Out}(K/H)|$.

Lemma 1.5. [12] Let $S = Sz(q)$ be Suzuki group where $q = 2^{2m+1}$ and $m \geq 1$. Then $\omega(S)$ consists of exactly all factors of 4, $q-1$, $q-\sqrt{2q}+1$, and $q+\sqrt{2q}+1$.

Lemma 1.6. [16] Let q, k, l be natural numbers. Then
(a) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$,
(b) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$
(c) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$
In particular, for every $q \geq 2$ and $k \geq 1$, the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

Results

In this section, we prove that the Suzuki groups are characterizable by their order and the largest order of elements. In fact, we prove that if G is a group with $|G| = |Sz(q)|$ and $k(G) = k(Sz(q))$, where $q-1$ or $q \pm \sqrt{2q} + 1$ is a prime number, then $G \cong Sz(q)$. We divide the proof to several lemmas. From now on, we denote the Suzuki group $Sz(q)$, where $q = 2^{2m+1}$, $m \geq 1$ by S and the number $q-1$ or $q \pm \sqrt{2q}+1$ by p. The group $Sz(q)$ is discoveres by M. Suzuki in [13], and its order is $q^2(q^2+1)(q-1)$. Also, it is the only simple non-abelian group of order prime to 3 [4].

Lemma 2.1. p is an isolated vertex of $\Gamma(G)$.
Proof. By Lemma 1.5, we have $k(S) = q + \sqrt{2q} + 1$. We prove that p is an isolated vertex of $\Gamma(G)$. If $q = 8$, then by Atlas [3], $Sz(8)$ has no element of order 13i, $i \geq 2$. So $k(S) = 13$ and p is an isolated vertex of $\Gamma(G)$. Now let $q > 8$ and p is not an isolated vertex of $\Gamma(G)$. So there is the natural number t such that $t \neq p$ and $tp \in \omega(G)$. Thus we deduce that $tp \geq 2p \geq 2(q - \sqrt{2q} + 1) > (q + \sqrt{2q} + 1)$ and hence $k(G) > q + \sqrt{2q} + 1$which is impossible. So we conclude that p is an isolated vertex of $\Gamma(G)$ and $t(G) \geq 2$.

Lemma 2.2. G is not a Frobenius group.
Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 1.1(a), $t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$. Since p is an isolated vertex of $\Gamma(G)$, we have **(i)** $|H| = \frac{|G|}{p}$ and $|K| = p$, or **(ii)** $|H| = p$ and $|K| = \frac{|G|}{p}$.

Assume that $|H| = \frac{|G|}{p}$ and $|K| = p$. Then Lemma 1.1(b) implies that $\frac{|G|}{p}$ divides $p - 1$ and hence $\frac{|G|}{p} \leq p - 1$ which is impossible. Therefore, the case $|H| = p$ and $|K| = \frac{|G|}{p}$ will be considered. Now Lemma 1.1(b) implies that p divides $\frac{|G|}{p} - 1$. We show that it is impossible. If $p = q - 1$, then $(q-1) \mid q^2(q^2+1) - 1$. So we deduce that $(q-1) \mid (q-1)(q^3+q^2+2q+2)+1$ which is impossible. If $p = q \pm \sqrt{2q} + 1$, then $(q \pm \sqrt{2q} + 1) \mid q^2(q \mp \sqrt{2q} + 1)(q-1) - 1$. So we deduce that $(q \pm \sqrt{2q}+1) \mid (q \pm \sqrt{2q}+1)(q^3 \mp 2\sqrt{2q}q^2 + 3q^2 - 4q \pm 4\sqrt{2q} - 4) + 3$. Hence $(q \pm \sqrt{2q}+1) \mid 3$ which is impossible.

Lemma 2.3. G is not a 2-Frobenius group.
Proof. Let G be a 2-Frobenius group. Then by Lemma 1.3, there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively, $t(G) = 2$, $\pi(G/K) \cup \pi(H) = \pi_1$, $\pi(K/H) = \pi_2$,G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|\mathrm{Aut}(K/H)|$. Since p is an isolated vertex of $\Gamma(G)$, we deduce $\pi_2 = \{p\}$ and $|K/H| = p$. If $p = q \pm \sqrt{2q} + 1$, then by Lemma 1.6, $(p-1, q-1) = 1$ and since $|G/K| \mid p - 1$, we deduce that $q - 1$ divides $|H|$. So $K/H \rtimes H_{q-1}$ is a Frobenius

group with kernel H_{q-1}. Hence Lemma 1.1(b) implies that $p \mid q-2$. We know that p is an odd prime. Thus $p \mid \frac{(q-2)}{2}$. Hence $2p \mid (q-2)$. So $2(q \pm \sqrt{2q}+1) \leq (q-2)$ which is impossible. If $p = q-1$, then the order of a Sylow 2-subgroup of G/K is at most 2 because $|G/K| \mid p-1$. So $\frac{q^2}{2}$ divides $|H|$. Hence $K/H \rtimes H_2$ is a Frobenius group with kernel H_2. Thus Lemma 1.1(b) implies that $p \mid |H_2|-1$.

If $|H_2| = \frac{q^2}{2}$, then $2^{2m+1}-1 \mid 2^{4m+1}-1$ which by lemma 1.6, is a contradiction. So we deduce that $|H_2| = q^2$ and $2 \nmid |G/K|$. Let $|G/K| = x$. Then $x \mid p-1$. On the other hand, $x \mid q^2+1$, hence $x \mid 5$ and $|H| = \frac{q^2(q^2+1)}{x}$. Now since K is a Frobenius group with kernel H, we deduce that $|K/H| \mid |H|-1$. So $q-1 \mid q^2(q^2+1)-x$. Thus $(q-1) \mid (q-1)(q^3+q^2+2q+2)-(x-2)$ which is a contradiction.

Lemma 2.4. The group G is isomorphic to the group S.

Proof. By Lemma 2.1, $t(G) \geq 2$. Thus G satisfy one of the statements of Lemma 1.4. Now Lemmas 2.2 and 2.3 imply that G satisfies only the statement (c). Hence G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups K/H is a non-abelian simple group. Since K/H is a non-abelian simple group and $3 \nmid |K/H|$, we deduce that $K/H \cong Sz(q')$, where $q' = 2^{2m'+1}$, $m' \geq 1$. We know that $H \trianglelefteq K \trianglelefteq G$, hence $2^{2m'+1} \leq 2^{2m+1}$. So $m' \leq m$. On the other hand, p is an isolated vertex of $\Gamma(G)$. Thus we deduce that $p \in \pi_2$ and hence $p \mid |K/H|$. So $q-\sqrt{2q}-1 \leq p \leq q'+\sqrt{2q'}+1 = k(K/H)$. Thus we deduce that $m \leq m'$ and $Sz(q') = S$. Now since $|K/H| = |S|$ and $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H = 1$ and $G = K \cong S$.

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References

1. Chen G.Y., He L.G., and Xu H.J. A new characterization of sporadic simple groups. *Ital. J. Pure Appl. Math.* **30**: 373-392 (2013).
2. Chen G.Y. On structure of Frobenius and 2-Frobenius groups. *J. Southwest China Normal Univ.* **20**: 485-487 (1995).
3. Conway J.H., Curtis R.T., Norton S.P., and Parker R.A. *Atlas of Finite Groups*. Clarendon, Oxford, (1985).
4. Glauberman G. Factorizations in local subgroups of finite groups. Regional Conference Series in Mathematics, No. 33. *American Mathematical Society, Providence, R.I.,* (1977).
5. Guo W. Structure theory for canonical classes of finite groups. Springer, Heidelberg, 191-248 (2015).
6. He L.G., and Chen G.Y. A new characterization of simple K_3-groups. *Commun. Algebra* **40**: 3903-3911 (2012).
7. He L.G., and Chen G.Y. A new characterization of $L_2(q)$ where $q = p^n < 125$. *Ital. J. Pure Appl. Math.* 125-134 (2011).
8. He L.G., and Chen G.Y. A new characterization of $L_3(q)$ $(q \leq 8)$ and $U_3(q)$ $(q \leq 11)$. *J. Southwest Univ. (Natur. Sci.)* **33**: 81-87 (2011).
9. Jiang Q., and Shao C. Characterization of some $L_2(q)$ by the largest element orders. *Math. Reports* **17**: 353-358 (2015).
10. Li J., Shi W., and Yu D. A characterization of some $PGL(2,q)$ by maximum element orders. *Bull. Korean Math. Soc.* **52**: 2025–2034 (2015).
11. Shi W.J. Pure Quantitative characterization of finite simple groups. *Fron. Math. China* **2**: 123-125 (2007).
12. Shi W.J. A characterization of Suzuki's simple groups. *Proc. Amer. Math. Soc.* **114**: 589-591 (1992).
13. Suzuki M. A new type of simple groups of finite order. *Proc. Nati. Acad. Sci. USA* **46**: 868-870 (1960).
14. Vasil'ev A.V., Grechkoseeva M.A., and Mazurov V.D. Characterization of the finite simple groups by spectrum and order. *Algebra Log.* **48**: 385-409 (2009).
15. Williams J.S. Prime graph components of finite groups. *J. Algebra* **69**: 487-513 (1981).
16. Zavarnitsine A.V. Recognition of the simple groups $L_3(q)$ by element orders. *J. Group Theory* **7**: 81-97 (2004).