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Admissibility of Linear Predictors of Finite Population Parameters under Reflected Normal Loss Function

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Abstract

One of the most important prediction problems in finite population is the prediction of a linear function of characteristic values of a finite population. In this paper the admissibility of linear predictors of an arbitrary linear function of characteristic values in a finite population under reflected normal loss function is considered. Under the super-population model, we obtain the conditions for which the linear predictors are admissible. Also, the risk of some admissible and inadmissible predictors are compared by a simulation study.

Keywords: Admissibility; Finite population; Linear predictor; Reflected normal loss function; Super-population model.

Introduction

A finite population is a collection of identifiable objects or elements. The students in a school, the households in a certain locality and etc. are examples of finite populations. A finite population of N units is denoted by index set $U=\{1,2,...,N\}$ and the characteristic value associated with i th unit in the population is denoted by y_i, $i=1,2,...,N.$ The vector $\underset{\sim}{\mathbf{y}}=(y_1,...y_N)^T$ is the unknown state of nature. In finite population, we usually want to estimate a linear function of characteristic values such as total value $Y=\sum_{i=1}^{N} y_i$, mean value $\bar{Y}=(\sum_{i=1}^{N} y_i)/N$, or generally $\gamma(\underset{\sim}{\mathbf{y}})=\sum_{i=1}^{N} p_i y_i$, where $p_i>0, i=1,...,N$ are known values. To estimate $\gamma(\underset{\sim}{\mathbf{y}})$, we first choose a sample $s\subset\{1,2,...,N\}$ with associated characteristic values $\underset{\sim}{\mathbf{y}}_S=\{y_k, k\in s\}$ by an arbitrary sampling design $p(s)$ ($p(s)>0$, and $\sum_{s\in S} p(s)=1$ where S is any subset of $U=\{1,2,...,N\}$). Then, from this sample we estimate $\gamma(\underset{\sim}{\mathbf{y}})$ by an estimator $\delta(\underset{\sim}{\mathbf{y}}_S)$.

The problem of estimation of an arbitrary linear function of the characteristic values $\gamma(\underset{\sim}{\mathbf{y}})=\sum_{i=1}^{N} p_i y_i$, in a finite population, has been considered and studied in the literature. One of the most interesting estimation problems is the admissibility of a given estimator. The problem of admissibility of an estimator of $\gamma(\underset{\sim}{\mathbf{y}})$ has been considered by many statisticians according to design-based and model-based approaches. In design-based approach (where we choose the sample s by a design

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$p(s)$), Godambe [7] proved that the Horvitz-Thompson [12] estimator is admissible in the class of linear unbiased estimators under the Squared Error Loss (SEL) function. Godambe and Joshi [9] extended Godambe's [7] result and proved that the Horvitz-Thompson estimator is admissible in the class of all unbiased estimators. For more details, see Xu [26], He and Xu [11], Hu et al. [13] and Peng et al. [20]. Also, the problem of Bayes estimation of $\gamma(\underset{\sim}{\mathbf{y}})$ in finite population is considered in the literature by many researchers. For more details, see Joshi [14], Mashayekhi [18], Karunamuni and Zhang [15], Ghosh and Sinha [10], Bansal and Aggarwal [3], Chen et al. [6], Si et al. [22] and Zangeneh and Little [27].

In model-based approach, the values $y_1,...,y_N$ in a population are considered as a realization of random variables $Y_1,...,Y_N$. The finite population may, therefore, be looked upon as a sample from a super-population. In this case, estimation of $\gamma(\underset{\sim}{\mathbf{y}})=\sum_{i=1}^{N} p_i y_i$ based on the sample s is known as prediction of a function of unobserved y's. Bolfarine [2] considered the prediction of the total of characteristic values $\gamma(\underset{\sim}{\mathbf{y}})=\sum_{i=1}^{N} y_i$, in a finite population under the LINEX loss function. He obtained the Bayes estimators and discussed

the admissibility of these estimators. Zou [28] obtained all admissible linear estimators of $\gamma(\underset{\sim}{\mathbf{y}})$ under the LINEX loss function. Zou et al. [29] found all admissible linear estimators of $\gamma(\underset{\sim}{\mathbf{y}})$ in the class of linear and all estimators under the SEL function.

In literature, the prediction of the unknown value $\gamma(\underset{\sim}{\mathbf{y}})$ is achieved only under SEL and LINEX loss functions. These loss functions are symmetric and asymmetric functions of $\Delta=\delta(\underset{\sim}{\mathbf{y}}_s)-\gamma(\underset{\sim}{\mathbf{y}})$, respectively. But both of these losses are unbounded and are not appropriate for prediction of $\gamma(\underset{\sim}{\mathbf{y}})$. As an alternative, Spiring [23] in response to this criticism, suggests the Reflected Normal Loss (RNL) function, which is defined by

$$L(\Delta)=k\left(1-e^{-\frac{\Delta^2}{2\gamma^2}}\right), \quad (1)$$

where k is the maximum loss and γ is a shape parameter (see also Spring and Yeung [24]). The RNL function is a symmetric and bounded function of Δ,

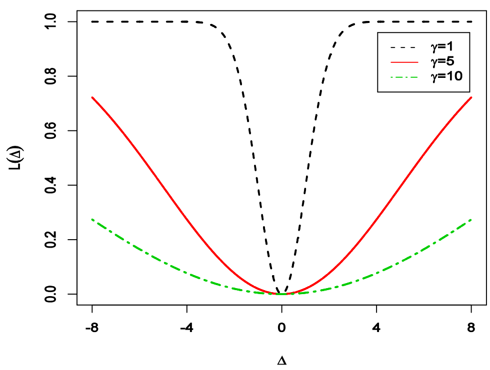

Figure 1. Plot of the RNL function for $k=1$ and certain values of γ

and is essentially a normal density flipped upside down, whence its name (Figure 1). Without loss of generality, we assume $k=1$.

According to our best knowledge, in the literature, there is no trace of obtaining the admissible predictor of $\gamma(\underset{\sim}{\mathbf{y}})$ under the RNL function. So, in this paper we consider the problem of admissible prediction of $\gamma(\underset{\sim}{\mathbf{y}})$ under the RNL function based on the super-population model.

This article is organized as follows. In the next section, we give some definitions, notations and preliminary results which are used throughout the paper. In the main result section, we provide sufficient conditions for admissibility of linear predictors of $\gamma(\underset{\sim}{\mathbf{y}})$ under the RNL function using the super-population model with known variance. Besides, there are some results illustrated by simulation study in the application section. Finally, a discussion is given in discussion section.

Preliminaries

In this section, we give some definitions and preliminary results which are used in the subsequent sections.

Finite population

In a finite population with index set $U=\{1,2,...,N\}$, the vector of characteristic values $\underset{\sim}{\mathbf{y}}=(y_1,...y_N)^T$ is the unknown state of nature and is assumed to belong to $\Theta=R^N$. A subset s of $\{1,2,...,N\}$ is called a sample and $n(s)$ denote the number of elements belonging to s. We consider a fixed number of sample, i.e., $n(s) = n$. A sample of size n is denoted by $\underset{\sim}{\mathbf{y}}_s = (y_{i_1},...,y_{i_n})^T$. In some prediction problems, it is necessary to predict a linear function of characteristic values of finite population. As in Zou et al. [29], we consider a general case, i.e., the prediction problem of an arbitrary linear function $\gamma(\underset{\sim}{\mathbf{y}}) = \sum_{k=1}^{N} p_k y_k$, $p_k > 0$.

Super-population model

We consider the model-based or super-population approach in which $\underset{\sim}{\mathbf{y}}$ is viewed as arising from a random sample of N units from some super-population having a probability density function (p.d.f) given by $f(y_i \mid \beta)$ where β may be either known or some unknown super-population parameter (Pfeffermann and Rao [19]). The model-based approach considers the values $y_1,...,y_N$ in the population as a realization of random variables $Y_1,...,Y_N$. The finite population may, therefore, be looked upon as a sample from a super-population distribution. After the sample has been observed, estimating $\gamma(\underset{\sim}{\mathbf{y}})$ reduces to predicting a function of unobserved Y 's. In Bayesian point of view, β is a random variable with density function $\pi(\beta)$, which is known as prior distribution.

Consider the following model for $\underset{\sim}{\mathbf{y}}$, (2)

$$\begin{cases} y_k = a_k\beta + b_k + \varepsilon_k, \quad \beta > 0 \\ E(\varepsilon_k) = 0,\ E(\varepsilon_k^2) = \sigma^2,\ E(\varepsilon_k\varepsilon_l) = 0,\ k \neq l, \end{cases} \qquad (3)$$

where $k = 1,2,...,N,\ a_k > 0$ and b_k are known constants and β is unknown parameter. This model is a basic and very useful one in survey sampling and was discussed in detail by Cassel et al. [4,5]. Also, Godambe [8] and Zou [28] considered this model.

Under the model (2) we assume $\varepsilon_k \sim N(0,\sigma^2),\ k = 1,..,N$. Hence, we have

(4)

$$f(y_k \mid \beta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_k - \beta a_k - b_k)^2},\ \beta \in \Re,\ a_k > 0,\ \sigma > 0,\ k = 1,...,N.$$

Assuming a normal prior for β with p.d.f.

(5)

$$\pi(\beta) = \frac{1}{\sqrt{2\pi}\tau} e^{-\frac{1}{2\tau^2}(\beta - \mu_0)^2},\quad \mu_0 \in \Re,\ \tau > 0\ are known$$

then the posterior p.d.f. of β given $\underset{\sim}{\mathbf{y}}_s$ is $N(\mu', w_\tau)$ with

$$\mu' = \frac{\sigma^2\mu_0}{\sigma^2 + \tau^2 d_s} + \frac{\tau^2 \sum_{k\in s} a_k(y_k - b_k)}{\sigma^2 + \tau^2 d_s},\quad and \quad w_\tau = \frac{\sigma^2\tau^2}{\sigma^2 + \tau^2 d_s},$$

where $d_s = \sum_{k\in s} a_k^2$.

Posterior risk function of a predictor under RNL function using normal model

Following Towhidi and Behboodian [25], the posterior risk of a predictor $\delta(\underset{\sim}{\mathbf{y}}_s)$ under the RNL function using normal model is given by

(6)

$$\rho(\pi, \delta(\underset{\sim}{\mathbf{y}}_s)) = 1 - E\left(e^{-\frac{1}{2\gamma^2}(\delta(\underset{\sim}{\mathbf{y}}_s) - \gamma(\mathbf{y}))^2} \mid \underset{\sim}{\mathbf{y}}_s\right) = 1 - \frac{\gamma}{\sqrt{\gamma^2 + w^*}} e^{-\frac{1}{2(w^* + \gamma^2)}(\delta(\underset{\sim}{\mathbf{y}}_s) - \mu^*)^2},$$

where

$$\mu^* = E(\gamma(\underset{\sim}{\mathbf{y}}) \mid \underset{\sim}{\mathbf{y}}_s),\ and\ w^* = V(\gamma(\underset{\sim}{\mathbf{y}}) \mid \underset{\sim}{\mathbf{y}}_s).$$

Hence, the posterior risk as a function of δ is minimized when $\delta^\pi(\underset{\sim}{\mathbf{y}}_s) = E(\gamma(\underset{\sim}{\mathbf{y}}) \mid \underset{\sim}{\mathbf{y}}_s) = \mu^*$, which is the same as Bayes predictor of $\gamma(\mathbf{y})$ under the SEL function. From (5), the posterior risk of $\delta^\pi(\underset{\sim}{\mathbf{y}}_s)$ is

$$\rho(\pi(\beta), \delta^\pi)) = 1 - \frac{\gamma}{\sqrt{\gamma^2 + w^*}}. \qquad (7)$$

Since the posterior risk does not depend on $\underset{\sim}{\mathbf{y}}_s$, therefore the Bayes risk of $\delta^\pi(\mathbf{y}_s)$ is

$$r^{å}(\pi, \delta^\pi(\underset{\sim}{\mathbf{y}}_s)) = 1 - \frac{\gamma}{\sqrt{\gamma^2 + w^*}}. \qquad (8)$$

In the next section, we obtain the admissible linear predictors of $\gamma(\underset{\sim}{\mathbf{y}}) = \sum_{k=1}^{N} p_k y_k$.

Results

In this section, we obtain the admissible predictor of $\gamma(\underset{\sim}{\mathbf{y}})=\sum_{k=1}^{N} p_k y_k$ under the RNL function, using model (2) with normal assumption and known σ^2. The main result is given in the following theorem.

Theorem 1

Using super-population model (2) with normal assumption and known σ^2, under the RNL function, the predictor $T(s)=\sum_{k\in s} w_{ks} y_k + w_{0s}$ with $w_{ks}=\lambda_s a_k + p_k \ (k\in s)$, of linear function $\gamma(\underset{\sim}{\mathbf{y}})$ is admissible in the class of all linear predictors, if one of the following two conditions is satisfied:

(i) $0\le \lambda_s < c_s/d_s$, where $c_s=\sum_{k\notin s} p_k a_k$ and $d_s=\sum_{k\in s} a_k^2$,

(ii) $\lambda_s = c_s/d_s$, and $w_{0s}=-(\frac{c_s}{d_s}\sum_{k\in s} a_k b_k + \sum_{k\notin s} p_k b_k)$.

Proof: Using the transformation $y_k - b_k \to y_k$, we need only to consider the case $b_k=0$, $k=1,\dots,N$. In this case, the condition (ii) becomes

$(ii)'$ $\lambda_s = c_s/d_s$ *and* $w_{0s}=0$.

After some calculations, using model (2) and under the RNL function, the risk function of the predictor $T(s)=\sum_{k\in s} w_{ks}y_k + w_{0s}$ of $\gamma(\underset{\sim}{\mathbf{y}})$ is given by

$$R(\gamma(\underset{\sim}{\mathbf{y}}),T(s)) = 1 - E\left(e^{-\frac{1}{2\gamma^2}(T(s)-\gamma(\underset{\sim}{\mathbf{y}}))^2}\right) = 1-\frac{\gamma}{\sqrt{\gamma^2+m}}e^{-\frac{\mu^2}{2(m+\gamma^2)}}, \quad (9)$$

where

$$\mu=\beta(\sum_{k\in s}(w_{ks}-p_k)a_k - \sum_{k\notin s} p_k a_k) + w_{0s} \quad and \quad m=\sigma^2(\sum_{k\in s}(w_{ks}-p_k)^2 + \sum_{k\notin s} p_k^2).$$

In order to prove the above theorem, we consider the following three cases.

(1) Assume $\lambda_s=0$. In this case $T(s)=\sum_{k\in s} p_k y_k + w_{0s}$, and using (8) we have

$$R(\gamma(\underset{\sim}{\mathbf{y}}),T(s)) = 1-\frac{\gamma}{\sqrt{\gamma^2+m_1}}e^{-\frac{\mu_1^2}{2(m_1+\gamma^2)}}, \quad (10)$$

where

$$\mu_1 = w_{0s}-\beta c_s \quad and \quad m_1=\sigma^2\sum_{k\notin s} p_k^2.$$

Now assume $T(s)$ is not admissible and is dominated by the linear predictor $\delta=\sum_{k\in s} w_{ks}^* y_k + w_{0s}^*$ with $w_{ks}^*=\lambda_s^* a_k + p_k$. Using (8), we have

$$R(\gamma(\underset{\sim}{\mathbf{y}}),\delta) = 1-\frac{\gamma}{\sqrt{\gamma^2+m_2}}e^{-\frac{\mu_2^2}{2(m_2+\gamma^2)}}, \quad (11)$$

where

$$\mu_2=\beta\lambda_s^* d_s - \beta c_s + w_{0s}^* \quad and \quad m_2=\sigma^2\lambda_s^{*2} d_s + \sigma^2\sum_{k\notin s} p_k^2.$$

Since $T(s)$ is dominated by δ, we have

$$R(\gamma(\underset{\sim}{\mathbf{y}}),\delta)\le R(\gamma(\underset{\sim}{\mathbf{y}}),T(s)) \quad \forall\beta,$$

which from (9) and (10) is equivalent to

$$\frac{\gamma}{\sqrt{\gamma^2+m_2}}e^{-\frac{\mu_2^2}{2(m_2+\gamma^2)}} \ge \frac{\gamma}{\sqrt{\gamma^2+m_1}}e^{-\frac{\mu_1^2}{2(m_1+\gamma^2)}} \quad \forall\beta. \quad (12)$$

Let $\beta=w_{0s}/c_s$ and therefore $\mu_1=0$, so from (11) we have

$$\frac{\gamma}{\sqrt{\gamma^2+m_2}}e^{-\frac{\mu_2^2}{2(m_2+\gamma^2)}} \ge \frac{\gamma}{\sqrt{\gamma^2+m_1}}. \quad (13)$$

Using the fact that $m_2\ge m_1$, we have

$$\frac{\gamma}{\sqrt{\gamma^2+m_2}}e^{-\frac{\mu_2^2}{2(m_2+\gamma^2)}} \le \frac{\gamma}{\sqrt{\gamma^2+m_1}}e^{-\frac{\mu_2^2}{2(m_2+\gamma^2)}} < \frac{\gamma}{\sqrt{\gamma^2+m_1}}. \quad (14)$$

The inequality (13) contradicts the inequality (12), so the predictor $T(s)$ of $\gamma(\underset{\sim}{\mathbf{y}})$ is admissible for $\lambda_s=0$.

(2) Assume $0<\lambda_s<c_s/d_s$. Let β have the prior distribution as in (4). The Bayes predictor of $\gamma(\underset{\sim}{\mathbf{y}})$ with respect to the normal prior (4) and under the RNL function could be obtained as

$$\delta^{\pi}(\underset{\sim}{\mathbf{y}}_s) = E(\gamma(\underset{\sim}{\mathbf{y}}) \mid \underset{\sim}{\mathbf{y}}_s) = \sum_{k\in s}(\frac{\tau^2 c_s}{\sigma^2+\tau^2 d_s}a_k + p_k)y_k + \frac{\sigma^2 c_s \mu_0}{\sigma^2+\tau^2 d_s}. \quad (15)$$

Therefore when $0<\lambda_s<c_s/d_s$, the predictor

$$T(s)=\sum_{k\in s} w_{ks}y_k + w_{0s} = \sum_{k\in s}(\lambda_s a_k + p_k)y_k + w_{0s}$$

is the Bayes predictor with respect to the prior distribution $N(\mu_0^*, \tau^{2*})$ where

$$\mu_0^* = \frac{w_{0s}}{c_s - \lambda_s d_s}, \ \tau^{2*} = \frac{\sigma^2 \lambda_s}{c_s - \lambda_s d_s}.$$

We can easily obtain the Bayes risk of $T(s)$ by using (7). After some calculations, we can show that the Bayes risk of $T(s)$ is finite. Now, since the loss function (1) is bowl-shaped, then $T(s)$ is the unique Bayes predictor and hence admissible.

(3) When $\lambda_s = c_s/d_s$ and $w_{0s}=0$, by using the limiting Bayes method (see Blyth [1] and Lehmann and Casella [17]), we can show that $T(s)$ is admissible. In fact, from (9), the risk function of the predictor $T(s)=\sum_{k\in s}(\frac{c_s}{d_s}a_k+p_k)y_k$ of $\gamma(\underset{\sim}{\mathbf{y}})$ is given by

$$R(\gamma(\underset{\sim}{\mathbf{y}}),T(s)) = 1-\frac{\gamma}{\sqrt{\gamma^2+m_3}} = r, \quad (16)$$

where

$$m_3 = \sigma^2((\frac{c_s}{d_s})^2 d_s + \sum_{k\notin s} p_k^2).$$

Suppose that the predictor $T(s)$ is dominated by δ^*, then

$$R(\gamma(\underset{\sim}{\mathbf{y}}),\delta^*) \le R(\gamma(\underset{\sim}{\mathbf{y}}),T(s)) \quad \textit{for all } \beta, \quad (17)$$

$$R(\gamma(\underset{\sim}{\mathbf{y}}),\delta^*) < R(\gamma(\underset{\sim}{\mathbf{y}}),T(s)) \quad \textit{for some } \beta_0. \quad (18)$$

Since $R(\gamma(\underset{\sim}{\mathbf{y}}),\delta^*)$ is a continuous function of β, then there exist an $\varepsilon>0$ and $\beta_1<\beta_2$ such that

$$R(\gamma(\underset{\sim}{\mathbf{y}}),\delta^*) < R(\gamma(\underset{\sim}{\mathbf{y}}),T(s)) - \varepsilon \quad \textit{for all } \ \beta_1<\beta<\beta_2. \quad (19)$$

Let δ_B be the Bayes predictor with respect to the prior distribution $N(0,\tau_B^2)$, and let $r^*(\delta_B)$ be the Bayes risk of δ_B. From (14), we have

$$\delta_B = \delta^{\pi}(\underset{\sim}{\mathbf{y}}_s) = \sum_{k\in s}(\frac{\tau_B^2 c_s}{\sigma^2+\tau_B^2 d_s}a_k + p_k)y_k. \quad (20)$$

From (7), the Bayes risk of δ_B could be obtained as

$$r^*(\delta_B) = 1-\frac{\gamma}{\sqrt{\gamma^2+w_B^*}}, \quad (21)$$

where

$$w_B^* = \sigma^2\left[\left(\frac{\tau_B^2 c_s}{\sigma^2+\tau_B^2 d_s}\right)^2 d_s + \sum_{k\notin s} p_k^2\right].$$

Let $r^*(\delta^*)$ be the Bayes risk of the predictor δ^* with respect to the prior distribution $N(0,\tau_B^2)$. Then from (15), (17), (18) and (20), we get

$$\frac{r-r^*(\delta^*)}{r-r^*(\delta_B)} > \frac{\int_{\beta_1}^{\beta_2}[R(\gamma(\underset{\sim}{\mathbf{y}}),T(s)) - R(\gamma(\underset{\sim}{\mathbf{y}}),\delta^*)]\frac{1}{\sqrt{2\pi}\tau_B}e^{-\frac{1}{2\tau_B^2}\beta^2}d\beta}{\frac{\gamma}{\sqrt{\sigma^2\left[\left(\frac{\tau_B^2 c_s}{\sigma^2+\tau_B^2 d_s}\right)^2 d_s + \sum_{k\notin s}p_k^2\right]+\gamma^2}} + \frac{\gamma}{\sqrt{\sigma^2\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \sum_{k\notin s}p_k^2\right]+\gamma^2}}}$$

$$> \frac{\frac{\varepsilon}{\sqrt{2\pi}\tau_B}\int_{\beta_1}^{\beta_2}e^{-\frac{1}{2\tau_B^2}\beta^2}d\beta}{\frac{\gamma}{\sqrt{\sigma^2\left[\left(\frac{\tau_B^2 c_s}{\sigma^2+\tau_B^2 d_s}\right)^2 d_s + \sum_{k\notin s}p_k^2\right]+\gamma^2}} + \frac{\gamma}{\sqrt{\sigma^2\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \sum_{k\notin s}p_k^2\right]+\gamma^2}}} \to +\infty$$

when $\tau_B \to +\infty$. Thus, if τ_B is sufficiently large, then $r^*(\delta^*) < r^*(\delta_B)$, which contradicts the fact that δ_B is the Bayes predictor with respect to the prior distribution $N(0,\tau_B^2)$. Therefore, $T(s)$ is admissible. Cases (1)-(3) complete the proof of the Theorem.

Remark 1

Following Lehmann [16] under the general loss function $L(\gamma(\underset{\sim}{\mathbf{y}}),\delta(\underset{\sim}{\mathbf{y}}_s))$, an estimator $\delta(\underset{\sim}{\mathbf{y}}_s)$ is risk unbiased estimator of $\gamma(\underset{\sim}{\mathbf{y}})$ if it satisfies

$$E_{\gamma}\left[L(\gamma(\underset{\sim}{\mathbf{y}}),\delta(\underset{\sim}{\mathbf{y}}_s))\right] \le E_{\gamma}\left[L(\gamma'(\underset{\sim}{\mathbf{y}}),\delta(\underset{\sim}{\mathbf{y}}_s))\right], \quad \forall \gamma(\underset{\sim}{\mathbf{y}}) \ne \gamma'(\underset{\sim}{\mathbf{y}}). \quad (21)$$

Under the SEL function, the above condition of risk unbiasedness reduces to the usual condition of unbiasedness, i.e., $E(\delta(\underset{\sim}{\mathbf{y}}_s)) = \gamma(\underset{\sim}{\mathbf{y}})$, and if an estimator $\delta(\underset{\sim}{\mathbf{y}}_s)$ is not unbiased, then its bias is given by $Bias(\delta(\underset{\sim}{\mathbf{y}}_s)) = E(\delta(\underset{\sim}{\mathbf{y}}_s)) - \gamma(\underset{\sim}{\mathbf{y}})$. So, the risk unbiased condition and bias of an estimator are depend on the loss function that we use. Under the RNL function, the risk unbiased condition (21) reduces to

$$E\left(\frac{1}{\gamma^2}(\delta(\underset{\sim}{\mathbf{y}}_s) - \gamma(\mathbf{y}))e^{-\frac{1}{2\gamma^2}(\delta(\underset{\sim}{\mathbf{y}}_s)-\gamma(\mathbf{y}))^2}\right) = 0 .$$

Therefore we can define the risk bias of an estimator $\delta(\underset{\sim}{\mathbf{y}}_s)$ of $\gamma(\underset{\sim}{\mathbf{y}})$ under the RNL function as

$$Risk\ \ Bias(\delta(\underset{\sim}{\mathbf{y}}_s)) = E\left(\frac{1}{\gamma^2}(\delta(\underset{\sim}{\mathbf{y}}_s) - \gamma(\mathbf{y}))e^{-\frac{1}{2\gamma^2}(\delta(\underset{\sim}{\mathbf{y}}_s)-\gamma(\mathbf{y}))^2}\right). \tag{22}$$

In what follows, we use (22) to compute the risk bias of the given estimators in a simulation study.

Comparison of the predictors using simulation data

In this section, we perform a simulation study to compare the predictors of $\overline{Y} = (\sum_{i=1}^{N} y_i)/N$ under RNL function. For generating a population of size $N = 1000$, we generate a random sample from normal distributions with different values of variance $\sigma^2 = 0.01, 0.1, 0.25, 0.5, 1$ and means $a_k\beta$, $k = 1,...,1000$, where $\beta = 3,4,5,7$ and $\underset{\sim}{a} = (a_1,...,a_{1000})$ is an arbitrary vector of positive elements. The population consist of these $N = 1000$ data. Now we extract samples of size $n = 5,10,15,20,50$ and compute predictors, risk function and risk bias of them. Repeat this tasks $B = 10^4$ times and calculate the estimated risk function and risk bias of the predictors as a comparative tool. The simulation study proceeds as follows:

1. Generate a sample with size 1000 from normal distributions: $N(a_k\beta, \sigma^2)$, $k = 1,...,1000$.

2. Use simulation data as a population and extract samples of size $n = 5,10,15,20,50$.

3. Calculate the predictors for each sample as follows:

$$\delta_1^a(\underset{\sim}{\mathbf{y}}_s) = \sum_{k\in s}(\frac{c_s}{2d_s}a_k + p_k)y_k,\quad \delta_2^a(\underset{\sim}{\mathbf{y}}_s) = \sum_{k\in s}(\frac{3c_s}{4d_s}a_k + p_k)y_k + w_{0s},$$

$$\delta_1^n(\underset{\sim}{\mathbf{y}}_s) = \sum_{k\in s}(-\frac{c_s}{d_s}a_k + p_k)y_k,\quad \delta_2^n(\underset{\sim}{\mathbf{y}}_s) = \sum_{k\in s}(2\frac{c_s}{d_s}a_k + p_k)y_k .$$

Where δ_1^a and δ_2^a are admissible predictors satisfying the conditions of Theorem 3.1 and the predictors δ_1^n and δ_2^n don't satisfying those conditions. We set $w_{0s} = -c_s$ in δ_2^a.

4. Repeat steps 2-3 $B = 10^4$ times and calculate the value of estimated risk function (ERF) and estimated absolute risk bias (EARB) of the predictors using the following formulas:

$$ERF = 1 - \frac{1}{B}\sum_{j=1}^{B}\exp\{-(\delta_{ij}^k - \gamma(\underset{\sim}{\mathbf{y}}))^2/2\gamma^2\},\quad \mathrm{i}=1,2,\quad k = a,n,$$

$$EARB = \left|\frac{1}{B}\sum_{j=1}^{B}(\delta_{ij}^k - \gamma(\underset{\sim}{\mathbf{y}}))\exp\{-(\delta_{ij}^k - \gamma(\underset{\sim}{\mathbf{y}}))^2/2\gamma^2\}/\gamma^2\right|,\quad i=1,2,\ k = a,n,$$

where $\gamma = 2$ and δ_{ij}^k is the predictor δ_i^k, $i = 1,2$, $k = a,n$ in j th repetition of sampling. Tables 1-5 present the estimated risk function and estimated risk bias (in parenthesis) of the predictors for different values for β and σ^2. From these tables we observe that δ_1^a and δ_2^a have the smallest risk among the four predictors and dominate δ_1^n and δ_2^n for all values of β and σ^2. So, δ_1^n and δ_2^n are inadmissible. Note that δ_1^a (δ_2^a) for small (large) values of β has smallest risk among these predictors. Also, the risk of all predictors increases as β increases. When σ^2 increases, the risk of inadmissible predictors δ_1^n and δ_2^n decreases and the risk of admissible predictors δ_1^a and δ_2^a have no patterns. But for large values of β ($\beta = 7$) the risk of all predictors decreases as σ^2 increases. Furthermore, in almost all cases, the risk of predictors decrease as the sample size n increases. The results of estimated risk bias of the predictors in Tables 1-5, show that δ_1^n and δ_2^n have smallest estimated risk bias among these predictors for almost all β and the risk bias of δ_1^n is close to zero. When σ^2 increases,

Table 1. Simulated risk function and estimated risk bias (in parenthesis) for some predictors of $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ for $\sigma^2 = 0.01$ and different values of ß.

	$\beta = 3$				$\beta = 4$			
n	δ_1^a	δ_2^a	δ_1^n	δ_2^n	δ_1^a	δ_2^a	δ_1^n	δ_2^n
5	0.1428431 (0.2378885)	0.1894101 (0.2625259)	0.9148669 (0.0944558)	0.4597269 (0.2991968)	0.2396994 (0.2813828)	0.2398510 (0.2813643)	0.9874875 (0.0185136)	0.6652028 (0.247265)
10	0.1415236 (0.2370966)	0.1877090 (0.2618272)	0.9127647 (0.09632349)	0.4563263 (0.2998343)	0.2376270 (0.2807716)	0.2377781 (0.2807872)	0.9869354 (0.01923796)	0.6616230 (0.2488868)
15	0.1401947 (0.2362677)	0.1859913 (0.2610655)	0.9106133 (0.09820985)	0.4530182 (0.3002461)	0.2355348 (0.280125)	0.2356786 (0.2801496)	0.9863571 (0.01998987)	0.6579932 (0.2503771)
20	0.1388338 (0.2354049)	0.1842195 (0.2602587)	0.9084416 (0.1000972)	0.4498847 (0.3005806)	0.2334017 (0.2794499)	0.2335183 (0.2794721)	0.985760 (0.02076099)	0.6544659 (0.2517787)
50	0.1310484 (0.2302795)	0.1741602 (0.2554332)	0.8942346 (0.1120913)	0.4296163 (0.302154)	0.2210316 (0.2752824)	0.2211499 (0.2753181)	0.9815972 (0.02600919)	0.6315701 (0.2602871)
	$\beta = 5$				$\beta = 7$			
5	0.3483347 (0.3014971)	0.2932344 (0.2943094)	0.9989370 (0.0019661)	0.8190645 (0.1670604)	0.5680374 (0.279817)	0.4045212 (0.3030775)	0.9999985 (0.0000003)	0.9649467 (0.0453183)
10	0.3455610 (0.301297)	0.2908026 (0.2939112)	0.998863 (0.0020930)	0.8161054 (0.169103)	0.5644279 (0.2807682)	0.4014696 (0.30317)	0.9999983 (0.0000004)	0.9638405 (0.0465601)
15	0.3427571 (0.3010653)	0.2883346 (0.2934564)	0.9987834 (0.0022284)	0.8130285 (0.1711256)	0.5607679 (0.2817032)	0.3983638 (0.3032171)	0.9999981 (0.0000004)	0.9626497 (0.0478674)
20	0.3399043 (0.3008114)	0.2857999 (0.2929672)	0.9986990 (0.0023712)	0.8099853 (0.173091)	0.5570451 (0.2826323)	0.3951813 (0.3032416)	0.9999978 (0.0000005)	0.9614380 (0.0491847)
50	0.3231801 (0.2990046)	0.2711817 (0.2898401)	0.9980577 (0.0034317)	0.7899758 (0.1854963)	0.5347707 (0.2877669)	0.3765925 (0.3030189)	0.9999952 (.0000011)	0.9530882 (0.0580150)

Table 2. Simulated risk function and estimated risk bias (in parenthesis) for some predictors of $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ for $\sigma^2 = 0.1$ and different values of ß.

	$\beta = 3$				$\beta = 4$			
n	δ_1^a	δ_2^a	δ_1^n	δ_2^n	δ_1^a	δ_2^a	δ_1^n	δ_2^n
5	0.1428778 (0.2373106)	0.1899330 (0.2615838)	0.9134266 (0.09546138)	0.457347 (0.2944187)	0.2395281 (0.2808586)	0.2401986 (0.2804178)	0.9871300 (0.0189325)	0.6604928 (0.2457796)
10	0.1414361 (0.236748)	0.1880036 (0.2613882)	0.9116745 (0.09715107)	0.4542422 (0.2975948)	0.2374198 (0.2804762)	0.2379909 (0.2803409)	0.9866810 (0.0195452)	0.6585625 (0.2484602)
15	0.1400460 (0.2359765)	0.1861740 (0.2607681)	0.9096459 (0.09896682)	0.4511625 (0.2987977)	0.2352894 (0.2798926)	0.2358063 (0.2798488)	0.9861356 (0.0202604)	0.6555643 (0.2502369)
20	0.1385680 (0.2350873)	0.1842029 (0.2599679)	0.9076021 (0.1007597)	0.4485780 (0.2995003)	0.2330346 (0.2792162)	0.2334499 (0.2791968)	0.9855674 (0.02099648)	0.6527223 (0.251646)
50	0.1307923 (0.2300419)	0.1741307 (0.2553004)	0.8934133 (0.1127423)	0.4283026 (0.3017492)	0.2207037 (0.2751161)	0.2210976 (0.275194)	0.9813986 (0.02624932)	0.6301796 (0.2604478)
	$\beta = 5$				$\beta = 7$			
5	0.3479880 (0.3010821)	0.2934087 (0.2933897)	0.9988914 (0.0020393)	0.8142886 (0.1681261)	0.5674899 (0.2796951)	0.4043767 (0.302291)	0.9999984 (0.0000004)	0.9628633 (0.047092)
10	0.3452591 (0.301082)	0.2909337 (0.2934726)	0.9988315 (0.0021441)	0.8132544 (0.1699556)	0.5640361 (0.2807386)	0.4014476 (0.302786)	0.9999982 (0.0000004)	0.9626741 (0.047586)
15	0.3424403 (0.3009101)	0.2884076 (0.293162)	0.9987561 (0.0022730)	0.8108557 (0.1718577)	0.5603976 (0.2817089)	0.3983350 (0.302961)	0.9999980 (0.0000005)	0.9617857 (0.048638)
20	0.3394697 (0.3006757)	0.2856833 (0.2927148)	0.9986750 (0.0024103)	0.8084076 (0.1736032)	0.5565816 (0.2826809)	0.3949835 (0.303051)	0.9999977 (0.0000005)	0.9607979 (0.049750)
50	0.3228072 (0.2989169)	0.2711084 (0.2897275)	0.9980305 (0.0034752)	0.7888290 (0.1859771)	0.5343881 (0.28782)	0.3764845 (0.302936)	0.9999951 (0.0000012)	0.9526440 (0.058417)

the estimated risk bias of admissible predictors δ_1^a and δ_2^a decreases and the estimated risk bias of inadmissible predictor δ_1^n increases. The estimated risk bias of δ_2^n for small values of $\beta(\beta = 3,4)$ decreases and for large values of $\beta(\beta = 5,6)$ increases as σ^2 increases. Also, when β increases,

Table 3. Simulated risk function and estimated risk bias (in parenthesis) for some predictors of $\bar{Y}=\frac{1}{N}\sum_{i=1}^{N}y_i$ for $\sigma^2=0.25$ and different values of ß.

	$\beta=3$				$\beta=4$			
n	δ_1^a	δ_2^a	δ_1^n	δ_2^n	δ_1^a	δ_2^a	δ_1^n	δ_2^n
5	0.1431960	0.1908865	0.9115639	0.4547678	0.2395511	0.2408678	0.9866324	0.6538721
	(0.2365118)	(0.260062)	(0.0966448)	(0.2865557)	(0.2800806)	(0.2788779)	(0.0194958)	(0.242833)
10	0.1415035	0.1885034	0.9104396	0.4521434	0.2373271	0.2383555	0.9863708	0.6546179
	(0.236301)	(0.2606631)	(0.0980132)	(0.2937791)	(0.2800624)	(0.2796023)	(0.0199069)	(0.2472884)
15	0.1400149	0.1864962	0.9086203	0.4493682	0.2351373	0.2360383	0.9858843	0.6526026
	(0.2356288)	(0.2602816)	(0.0997134)	(0.2962802)	(0.2795864)	(0.2793543)	(0.0205573)	(0.2495729)
20	0.1384136	0.1843136	0.9067399	0.4473081	0.2327655	0.2334847	0.9853569	0.6505791
	(0.2347427)	(0.2595478)	(0.1013973)	(0.2976293)	(0.2789364)	(0.278786)	(0.0212460)	(0.2511275)
50	0.1306059	0.1741591	0.8926575	0.4271461	0.2204447	0.2210941	0.9812093	0.6287511
	(0.2298087)	(0.2551177)	(0.1133211)	(0.3010142)	(0.2749404)	(0.2750165)	(0.0264738)	(0.2603998)
	$\beta=5$				$\beta=7$			
5	0.3477413	0.2937951	0.9988243	0.8071225	0.5668848	0.4042373	0.9999982	0.9594932
	(0.300417)	(0.2918832)	(0.0021444)	(0.1692659)	(0.2794111)	(0.3009846)	(0.0000004)	(0.0498086)
10	0.3450271	0.2911629	0.9987910	0.8092747	0.5636358	0.4014227	0.9999981	0.9609117
	(0.3007459)	(0.2927455)	(0.0022086)	(0.1708404)	(0.2806239)	(0.302147)	(0.0000004)	(0.0490660)
15	0.3421882	0.2885495	0.9987235	0.8079684	0.5600385	0.3983081	0.9999979	0.9605356
	(0.3006754)	(0.292676)	(0.0023251)	(0.1725871)	(0.2816531)	(0.3025351)	(0.0000005)	(0.0497022)
20	0.3391139	0.2856449	0.9986474	0.8062981	0.5561552	0.3948152	0.9999977	0.9598706
	(0.3004834)	(0.2923254)	(0.0024543)	(0.1741154)	(0.2826762)	(0.3027352)	(0.0000005)	(0.0505331)
50	0.3224979	0.2710743	0.9980038	0.7875509	0.5340508	0.3763964	0.9999950	0.9520992
	(0.2988092)	(0.2895601)	(0.0035174)	(0.1863958)	(0.2878439)	(0.3028006)	(0.0000012)	(0.0588828)

Table 4. Simulated risk function and estimated risk bias (in parenthesis) for some predictors of $\bar{Y}=\frac{1}{N}\sum_{i=1}^{N}y_i$ for $\sigma^2=0.5$ and different values of ß.

	$\beta=3$				$\beta=4$			
n	δ_1^a	δ_2^a	δ_1^n	δ_2^n	δ_1^a	δ_2^a	δ_1^n	δ_2^n
5	0.1438719	0.1925181	0.9087605	0.4518238	0.2397642	0.2420345	0.9858453	0.6442187
	(0.2352783)	(0.2575772)	(0.0983138)	(0.2739971)	(0.2788423)	(0.2763559)	(0.0203631)	(0.2375071)
10	0.1417358	0.1893409	0.9087065	0.4495806	0.2373152	0.2389694	0.9859136	0.6488739
	(0.2356343)	(0.2594646)	(0.0991527)	(0.2874868)	(0.2794181)	(0.2783803)	(0.0204272)	(0.2450213)
15	0.1400849	0.1870427	0.9072388	0.4471855	0.2350284	0.2364358	0.9855295	0.6483589
	(0.2351272)	(0.2594784)	(0.1006654)	(0.2920719)	(0.279122)	(0.2785365)	(0.0209668)	(0.2481942)
20	0.1383187	0.1845760	0.9056033	0.4457451	0.2325097	0.2336265	0.9850668	0.6474810
	(0.2342727)	(0.2588856)	(0.1021957)	(0.2944965)	(0.278532)	(0.2781296)	(0.0215825)	(0.2500799)
50	0.1304306	0.1742500	0.8917417	0.4258083	0.2201745	0.2211352	0.9809731	0.6268782
	(0.2295118)	(0.2548351)	(0.1140009)	(0.2997578)	(0.2747046)	(0.2747375)	(0.0267495)	(0.2601379)
	$\beta=5$				$\beta=7$			
5	0.3475190	0.2944979	0.9987137	0.7960637	0.5660530	0.4040736	0.9999979	0.9538439
	(0.2993266)	(0.2894084)	(0.0023149)	(0.1703516)	(0.2788919)	(0.2988239)	(0.0000005)	(0.0540443)
10	0.3447938	0.2915527	0.9987290	0.8031820	0.5631117	0.4013911	0.9999979	0.9580473
	(0.3001991)	(0.2915412)	(0.0023058)	(0.1718874)	(0.2803959)	(0.301086)	(0.0000005)	(0.0513656)
15	0.3419238	0.2887978	0.9986757	0.8036147	0.5595856	0.3982763	0.9999978	0.9585439
	(0.300298)	(0.2918708)	(0.0024004)	(0.1734675)	(0.2815232)	(0.3018265)	(0.0000005)	(0.0513344)
20	0.3387285	0.2856690	0.9986082	0.8031000	0.5556393	0.3946264	0.9999975	0.9583951
	(0.3001821)	(0.2916951)	(0.0025163)	(0.1747443)	(0.28262)	(0.3022078)	(0.0000006)	(0.0517403)
50	0.3221577	0.2710670	0.9979696	0.7857791	0.5336573	0.3763014	0.9999948	0.9512989
	(0.2986512)	(0.2892926)	(0.0035709)	(0.1868721)	(0.2878474)	(0.3025762)	(0.0000012)	(0.0595426)

the estimated risk of δ_1^a (δ_1^n and δ_2^n) increases (decreases) and the estimated risk bias of δ_2^a first increases and then decreases. Note that when the sample size n increases, the risk bias of all predictors increases, this phenomenon can be occurred in survey sampling as explained by Roxy et al. (2008, p. 34).

Table 5. Simulated risk function and estimated risk bias (in parenthesis) for some predictors of $\bar{Y}=\frac{1}{N}\sum_{i=1}^{N} y_i$ for $\sigma^2=1$ and different values of ß.

	$\beta=3$				$\beta=4$			
n	δ_1^a	δ_2^a	δ_1^n	δ_2^n	δ_1^a	δ_2^a	δ_1^n	δ_2^n
5	0.1453986 (0.2329472)	0.1958204 (0.2527456)	0.9035317 (0.1011986)	0.4489218 (0.2510127)	0.2404067 (0.2764523)	0.2444315 (0.2714378)	0.9842873 (0.0220192)	0.6284180 (0.2262291)
10	0.1423471 (0.2343985)	0.1910187 (0.2570984)	0.9056365 (0.1010471)	0.4460262 (0.2753517)	0.2374674 (0.2781888)	0.2402054 (0.2759649)	0.9850619 (0.0213706)	0.6389190 (0.2399983)
15	0.1403741 (0.2342214)	0.1881463 (0.2578898)	0.9048749 (0.1022019)	0.4440185 (0.283797)	0.2349886 (0.2782512)	0.2372445 (0.2769165)	0.9848926 (0.0216841)	0.6409590 (0.2450406)
20	0.1383277 (0.2334614)	0.1851957 (0.257612)	0.9036960 (0.1034642)	0.4434148 (0.2883003)	0.2322346 (0.2777999)	0.2340133 (0.276856)	0.9845573 (0.0221599)	0.6419991 (0.2477229)
50	0.1302455 (0.2290303)	0.1744851 (0.2542976)	0.8903299 (0.1150108)	0.4238779 (0.2972178)	0.2198320 (0.2743031)	0.2212751 (0.2742008)	0.9805965 (0.0271810)	0.6237774 (0.2593818)
	$\beta=5$				$\beta=7$			
5	0.3473130 (0.2971787)	0.2959891 (0.2845679)	0.9984814 (0.0026633)	0.7763497 (0.1704702)	0.5646168 (0.2777972)	0.4038644 (0.2945678)	0.9999970 (0.0000007)	0.9423839 (0.0616011)
10	0.3445174 (0.2991241)	0.2923454 (0.2891578)	0.9986084 (0.0024914)	0.7919922 (0.1731207)	0.5622416 (0.2798941)	0.4013488 (0.2989799)	0.9999975 (0.0000006)	0.9523439 (0.0556368)
15	0.3415867 (0.2995611)	0.2893108 (0.2902734)	0.9985867 (0.0025386)	0.7956159 (0.1746542)	0.5588594 (0.281218)	0.3982327 (0.3004161)	0.9999975 (0.0000006)	0.9546372 (0.054375)
20	0.3382125 (0.2996033)	0.2858266 (0.2904613)	0.9985367 (0.0026274)	0.7971744 (0.1756342)	0.5548469 (0.2824482)	0.3943635 (0.301156)	0.9999973 (0.0000006)	0.9555095 (0.0540055)
50	0.3216924 (0.2983619)	0.2711133 (0.2887724)	0.9979135 (0.0036575)	0.7826883 (0.1875361)	0.5330770 (0.2878098)	0.3761753 (0.3021298)	0.9999946 (0.0000013)	0.9498255 (0.0607149)

Discussion

In this paper we obtain sufficient condition of admissibility of linear predictors of $\gamma(\underset{\sim}{\mathbf{y}})=\sum_{k=1}^{N} p_k y_k$ in the class of all linear predictors under the RNL function. So, we could find admissible linear predictors of the total value $Y=\sum_{i=1}^{N} y_i$ and the mean value $\bar{Y}=(\sum_{i=1}^{N} y_i)/N$ of a finite population. Further research is needed to find the necessity condition of admissibility of linear predictors of $\gamma(\underset{\sim}{\mathbf{y}})$ under the RNL function. We perform a simulation study to compare the predictors of $\bar{Y}=(\sum_{i=1}^{N} y_i)/N$ that satisfy and do not satisfy the conditions of Theorem 3.1 under the RNL function. From Tables 1-5, for simulated data we observe that δ_1^a and δ_2^a have the smallest risks among the four predictors being considered and dominate δ_1^n and δ_2^n for all values of β and σ^2. So, δ_1^n and δ_2^n are inadmissible.

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