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Derivations of the Algebra of Sections of Superalgebra Bundles

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Abstract

In this paper we review the concepts of the superalgebra, superderivation and some properties of them. We will define algebraic and differential superderivations on a superalgebra and will prove some theorems about them, Then we consider a superalgebra bundle, that is an algebra bundle which its fibers are superalgebras and then characterize the superderivations of the algebra of sections of that bundle.

Keywords: Superalgebra; Superderivation; Algebraic superderivation; Differential superderivation.

Introduction

Replacing $C^{¥}(M)$ by $C^{\infty}(M, A)$, where A is an arbitrary algebra, is the starting point of noncommutative geometry [2], [5]. Many investigations about derivations on different kind of algebras have been done in different fields of mathematics such as algebra and analysis [4]. But superalgebras are very important in different fields of theoretical sciences such as physics [1]. So in this paper, we study the noncommutative differential geometry of the superalgebra of sections of a superalgebra bundle. In fact, we consider a superalgebra bundle Λ over a manifold M. We know the derivations on $C^{¥}(M)$ as vector fields. So we try to characterize superderivations on $\Gamma(\Lambda)$ similarly. In fact the superderivations on $\Gamma(\Lambda)$ can be considered as new vector fields on M which are sections of a supervector bundle.

In section 2, we review some essential facts about superalgebras and superderivations.

In section 3, for an arbitrary superalgebra bundle Λ over M, we describe all superderivations on $\Gamma(\Lambda)$ and construct a supervector bundle on M that its sections naturally correspond to superderivations on $\Gamma(\Lambda)$.

Definition 1.1 A superalgebra *A*, sometimes also called a Z_2-graded algebra, is a vector superspace $A = A_0 + A_1$ equipped with a bilinear multiplication satisfying $A_i A_j \subseteq A_{i+j}$ for $i, j \in Z_2$. The parity of a homogeneous element $a \in A_i$ is denoted by $|a| = i, i \in Z_2$. An element in A_0 is called even, while an element in A_1 is called odd.

Definition 1.2 A Lie superalgebra is a superalgebra $g = g_0 + g_1$ with bilinear multiplication $[.,.]$ satisfying the following two axioms, for homogeneous elements $a, b, c \in g$,

•Skew-super symmetry: $[a,b] = -(-1)^{|a|.|b|}[b,a]$.

•Super Jacobi identity

$$[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]].$$

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Example 1.3 Every graded associative algebra, with the following bracket, that is defiened for homogeneous elements and then is generalized bilinearly on the whole of the algebra, can be considered as a super Lie algebra:

$$[x,y]=xy-(-1)^{|x|.|y|}yx.$$

Definition 1.4 The supercenter of a super algebra is denoted by $Z_s(A)$, and is defined as a subalgebra of A, that its elements are as follows: a homogeneous element a is an element of $Z_s(A)$, if for every homogeneous element x we have

$ax=(-1)^{|x|.|a|}xa$, and an arbitrary element of A is in $Z_s(A)$ if its homogeneous (even and odd) parts are in $Z_s(A)$.

Definition 1.5 A linear map D from A to A, for $s\in Z_2$, is called a derivation of degree s if it satisfies that $D(ab)=D(a)b+(-1)^{s|a|}aD(b), a,b\in A$

In the above definition, D also is called a superderivation that s is called the parity of D and is denoted by $|D|$.

Example 1.6 For a homogeneous element a, the map $\delta_a:A\to A$ that is defined by $\delta_a(x)=-ax-(-1)^{|a|.|x|}xa$ is a superderivation with parity $|a|$. We call this the inner superderivation.

Example 1.7 The space of even (odd) superderivations on A is a vector subspace of $L(A)$ (the space of linear functions from A to A). We denote the direct sum of these two spaces by $Der_s(A)$ and call that the space of superderivations. $Der_s(A)$ with the following bracket is a Lie superalgebra:

$$[D_1,D_2]=D_1oD_2-(-1)^{|D_1|.|D_2|}D_2oD_1$$

where D_1,D_2 are homogeneous superderivaions.

Proposition 1.8 The supercenter is invariant under superderivations.

Proof. Let $b\in Z_s(A), a\in A$ and $D\in Der_s(A)$ be homogeneous elements. Then

$$\begin{aligned}
D(b)a &= D(ba)-(-1)^{|b||D|}bD(a)\\
&= D((-1)^{|a||b|}ab)-(-1)^{|b||D|}((-1)^{|b|(|D|+|a|)}D(a)b)\\
&= (-1)^{|b||a|}D(ab)-(-1)^{|a||b|}D(a)b\\
&= (-1)^{|b||a|}D(ab)-(-1)^{|b||a|}(D(ab)-(-1)^{|D||a|}aD(b))\\
&= (-1)^{|a|(|b|+|D|)}aD(b)\\
&= (-1)^{|a|(|D(b)|}aD(b)
\end{aligned}$$

Definition 1.9 For every vector field $X\in\chi M$, of smooth manifold M the Lie derivation along X on the functions in $C^\infty(M,A)$ is denoted by L_X and is defined as follows:

For constants functions f we have $L_X(f)=0$ and for functions in the form of $f=f_1e_1+\cdots+f_ke_k$ where $\forall i, f_i\in C^\infty(M)$ and $\{e_1,\cdots e_k\}$ is a basis for superalgebra A, we have

$$L_X(f)=X(f_1)e_1+\cdots+X(f_k)e_k.$$

Definition 1.10 The vector bundle (E, π, M) is called a supervector bundle(graded vector bundle), if its fibers are supervector spaces and the set of even (odd) vectors make a vector subbundle. Then E can be written as $E=E_0ÅE_1$ such that E_0 and E_1 are called even and odd subbundles.

Definition 1.11 In a supervector bundle (E, π, M), the sections that their values lies in even(odd) part are called even(odd) sections. The set of even (odd) sections are called homogeneous sections. With these definitions the set of sections of E(denoted by Γ(E)) is a supervector space that can be written as $G(E)=G(E)_0ÅG(E)_1$ where $G(E)_0=G(E_0)$ and $G(E)_1=G(E_1)$.

Definition 1.12 A supervector bundle (E, π, M) that it's fibers are superalgebras is called a superalgebra (graded) bundle

Results

From now on, we assume that M is a smooth manifold, A is a unital Z_2-graded algebra that has finite dimension and Λ is a superalgebra bundle on M with fibers isomorphic to A. By pointwise addition and multiplication the space of sections of Λ is an associative unital algebra. Since A is unital, $C^{\infty}(\mathrm{M})$ (smooth real valued functions on M) can be considered as a subalgebra of $\Gamma(\Lambda)$ (the space of sections of bundle Λ). The purpose of this article is to characterize the superderivations on the superalgebra $\Gamma(\Lambda)$.

Definition 2.1 set For each $x \in M$

$$Der_s(\Lambda) = \bigcup_{x\in \mathrm{M}} Der_s(\Lambda_x), \qquad \mathrm{Z}_s(\Lambda) = \bigcup_{x\in \mathrm{M}} \mathrm{Z}_s(\Lambda_x)$$

For each x∈M, Λ_x is a superalgebra isomorphic to A, so $Der_s(\Lambda_x)$ is a Lie superalgebra, isomorphic to $Der_s(A)$, and $Z_s(\Lambda_x)$ is a commutative superalgebra isomorphic to $Z_s(A)$.

Definition 2.2 for each section $q \in \Gamma(Der_s(\Lambda))$ we can define the linear function $D_q : \Gamma(\Lambda) \to \Gamma(\Lambda)$ as follows:

$$D_q(s)(x) = q(x)(s(x)), x \in \mathrm{M}, s \in \Gamma(\Lambda)$$

straightforward computations show that D_q is a superderivation on $\Gamma(\Lambda)$. If the values of q are homogenuos with fixed parities, then the parity of D_q is the same as parity of q(x).

Definition 2.3 A superderivation on $\Gamma(\Lambda)$ is called algebraic whenever it maps C¥(M) to the zero.

Example 2.4 D_q in last definition is an algebraic superderivation.

Theorem 2.5 Every algebraic superderivation can be written in the form of D_q for some smooth $q \in \Gamma(Der_s(\Lambda))$.

Proof. Let D be an algebraic superderivation on $\Gamma(\Lambda)$. For every $f \in C^{\infty}(\mathrm{M})$ and $S \in \Gamma(\Lambda)$, we have :

$$D(fS) = D(f)S + (-1)^{|D||f.1|} fD(S) = fD(S)$$

So, D is tensorial and for some section $Q \in \Gamma(L(A))$ we have $D(S)(x) = Q_x(S(x))$.

Since D is a superderivation, it implies that, Q_x is a superderivation on Λ_x so $Q \in \Gamma(Der_s(\Lambda))$ and $D = D_Q$.

In the following, some concepts about connections will be used from [6].

Definition 2.6 A connection ∇ on a superalgebra bundle Λ is called compatible superconnection if

$$\nabla_X(S_1S_2) = (\nabla_X S_1)S_2 + (-1)^{|\nabla_X S_1||S_1|}(\nabla_X S_2), \forall X \in \chi\mathrm{M}, S_1, S_2 \in \Gamma(\Lambda)$$

and $\tilde{N}_X$,S_1,S_2 are homogenous.

Corollary 2.7 If ∇ is compatible superconnection, then the $\tilde{N}_X$ is a superderivation with parity $|\tilde{N}_X|$ on $\Gamma(\Lambda)$.

Example 2.8 If A be a superalgebra and M a smooth manifold, then the algebra of sections of the trivial superalgebra bundle $M \times A$ is isomorphic to $C^{\infty}(\mathrm{M}, A)$. The Lie derivation along the vector field X of M, is a compatible superconnection on the bundle $M \times A$ ($\tilde{N}_X$=L_x is an even superderivation on $C^{\infty}(\mathrm{M}, A)$, that is called the trivial superconnection of that bundle.

Proposition 2.9 The superalgebra bundle Λ admits a compatible superconnection and the space of compatible superconnections is an affine space whose underlying vector space is $\mathrm{A}^1(\mathrm{M}, Der_s(\Lambda))$ (the set of 1−forms with values in $Der_s(\Lambda)$).

Proof. We know that every vector bundle has trivializations. By application of last example, we can find a campatible superconnection for every trivialization. So by using partition of unity for bundle and gluing lacal compatible superconnections to each other, we find a campatible superconnection for Λ. Clearly, difference of any two compatible superconnection, is a $Der_s(\Lambda)$-valued 1-form.

We have a one to one correspondance between the space of superderivations generated by $h\nabla_X, h \in \Gamma(Z_s(\Lambda))$ where $\tilde{N}_X$ is a compatible superconnection on Λ, and the space of sections of supervector bundle $\mathrm{TM} \otimes Z_s(\Lambda)$, as follows: this correspondance maps the $X = X_1 \otimes h_1 + \cdots + X_k \otimes h_k$ to the $\nabla X = h_1 \nabla_{X_1} + \cdots h_k \nabla_{X_k}$, where $X_i \in \chi M, h \in \Gamma(Z_s(\Lambda))$

Proposition 2.10 The intersection of the space of superderivations ∇_X and the space of algebraic superderivations is the set of zero superderivation.

Lemma 2.11 If ∇ is a compatible superconnection on trivial superalgebra bundle $M \times A$, then every superderivation on $C^\infty(M, A)$ is uniquely the sum of an algebraic superderivation and a superderivation ∇_X, where $X \in \Gamma(TM \otimes (M \times Z_s(A)))$.

Proof. Suppose D is a superderivation on $C^\infty(M, A)$ .Every $f \in C^\infty(M)$ is in the supercenter of $C^\infty(M, A)$, so $D(f)$ is a $Z_s(A)$-valued function. By chossing a basis $\{e_1, \ldots, e_k\}$ for $Z_s(A)$, we have $D(f) = D_1(f)e_1 + \cdots + D_k(f)e_k$ where $D_1, \ldots, D_k$ are derivations on $C^\infty(M)$ so, there exist $X_1, \cdots, X_k \in \chi M$ such that $D_1 = X_1, \ldots, D_k = X_k$. Now, by setting $X = X_1 \otimes e_1 + \cdots + X_k \otimes e_k$, we see that the superderivations D and L_X have the same value on real valued functions, so $D' = D - L_X$ will be constant zero on real valued functions, so it is an algebraic superderivation. Now we have the unique decomposition $D = D' + L_X$ for D.

Theorem 2.12 Let ∇ be a compatible superconnection on the Λ, then every superderivation on $\Gamma(\Lambda)$ can be written uniquely as the sum of an algebraic superderivation and a superderivation ∇_X, where $X \in \Gamma(TM \otimes (M \times Z_s(A))$.

Proof. Suppose D is a superderivation on $\Gamma(\Lambda)$. Let U_i be an open covering of M such that for each i, Λ has a trivialization on U_i. For each i, indeed Λ_i, the resriction of Λ to U_i, is a trivial superalgebra bundle.

Restricting D to the $\Gamma(\Lambda_i)$ yields a superderivation on $\Gamma(\Lambda_i)$ that we denote that by D_i Also we can restrict ∇ to $\Gamma(\Lambda_i)$ and denote this restriction by ∇^i that is compatible superconnection on Λ_i Now, For each i, we have an algebraic superderivation D'_i and a $X_i \in \Gamma(TM \otimes (M \times Z_s(\Lambda_i))$ such that $D_i = D'_i + \nabla^i_{X_i}$. If for some index i, j we have $U_i \bigcap U_j \neq \phi$, because of the uniqueness of the decomposition of superderivations on trivial bundles, we have equality of algebraic superderivations D'_i and D'_j, and sections X_i and X_j on $U_i \bigcap U_j$. So, the family of algebraic superderivations $\{D'_i\}$ and the family of sections $\{X_i\}$ by the application of partition of unity define an algebraic superderivation D' on $\Gamma(\Lambda)$ and a section X of $TM \otimes Z_s(\Lambda)$ such that $D = D' + \nabla_X$.

Discussion

One example of superalgebra bundles is the graded algebra of a smooth manifold that its derivations has been characterized in [3] and agrees with this paper.

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