Journal of Sciences, Islamic Republic of Iran 28(4): 359 - 367 (2017) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

A Compact Scheme for a Partial Integro-Differential Equation with Weakly Singular Kernel

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Received: 28 July 2016 / Revised: 15 March 2017 / Accepted: 13 June 2017

Abstract

Compact finite difference scheme is applied for a partial integro-differential equation with a weakly singular kernel. The product trapezoidal method is applied for discretization of the integral term. The order of accuracy in space and time is $O(h^4, k^{2-\alpha})$, where $0<\alpha<1$. Stability and convergence in L_2 norm are discussed through energy method. Numerical examples are provided to confirm the theoretical prediction and to show that the combination of the compact finite difference approximation and product trapezoidal method give an efficient method for solving a partial integro-differential equation.

Keywords: Compact finite difference; Partial integro-differential equation; Product trapezoidal method; Stability; Convergence.

Introduction

This study is focused on the investigation of a compact difference method for the following partial integro-differential equation

$$u_t = \mu(u_{xx} + \beta u_x) + \int_0^t (s-t)^{-\alpha}(u_{xx} + \beta u_x)ds, \qquad 0 < x < 1,\ t \ge 0, \tag{1}$$

where $\mu \ge 0$, $0 < \alpha < 1$, and β are real constants, with initial and boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x), \quad 0 \le x \le 1, \\ &u(0,t) = u(1,t) = 0, \quad t \ge 0. \end{aligned} \tag{2}$$

The partial integro-differential equations arise in a wide range of disciplines including physics, chemistry, and engineering. Specific examples of our interest here include modeling of wave propagation which involves viscoelastic forces, heat conduction in materials with memory and anomalous diffusion processes [1-3].

The standard and compact techniques in finite difference methods are usually used to obtain the numerical solutions of differential equations. These methods appear to compete with both the finite element and the spectral methods. The latter is preferred to the other two methods whenever the solution is highly regular and the geometric dimension of the domain becomes large. Spectral methods have been studied by variational techniques, to point out the dependence of the approximation error (for instance in the L_2 norm, or in the energy norm) on the regularity of the solution and the discretization parameter. Indeed, the solution is not often infinitely differentiable [4]. However, using fewer nodes per level, our proposed method can solve algebraic system whose coefficient matrix is of tridiagonal type. On the other hand, discretization of integral term to be used in equation (1), shall not contradict difference schem of derivaties term in that

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equation and shall result in the formation of tridiagonal matrix. Therefore, the combination of the compact finite difference approximation and product trapezoidal method gives an efficient method for solving the partial integro-differential equation (1), and would help us accomplish our goal.

A number of people have studied the integro-differential equations [5, 6], however, considerable works on numerical solutions of partial integro-differential equations have not been carried out. Lopez-Marcos studied the nonlinear partial integro-differential equation; he used one order full discrete difference scheme and a convolution quadrature for approximating the integral term [7]. Xu considered backward Euler method in time direction for a parabolic integro-differential equation and proved the stability and convergent properties of time discretizations [8, 9], Singh, et al. analyzed an efficient matrix method based on shifted Legendre polynomials for the solution of non-linear volterra singular partial integro-differential equations [10]. Fakhar-Izadi and Dehghan applied the spectral method for the partial integro-differential equations with a weakly singular kernel on irregular domains [11].

Tang presented finite difference scheme for Eq. (1) with $\beta = 0$, and $\alpha = \frac{1}{2}$ [12], Chen and Xu worked on theoretical analysis of compact difference scheme for an evolution equation with a weakly singular kernel with the truncation error of order $\frac{3}{2}$ in time and 4 in space, and the convergence and stability of their method was proved [13], and Luo, et al. considered compact finite difference scheme for Eq. (1) with $\beta = 0$ and $\alpha = \frac{1}{2}$ [14]. In the present study, the researchers attempt to give a compact difference scheme for Eq. (1) and prove that the compact difference scheme is stable and convergent in L_2 norm; moreover, the order of convergence will be proved to be $O(h^4, k^{2-\alpha})$.

This study is presented in the following sections: in Section 1, the product trapezoidal method and a fourth-order compact finite difference scheme for discretization of spatial derivatives are introduced. Results of section 1 are applied for discretization of Eq. (1) and product trapezoidal method to approximate the integral term of Eq. (1). Stability analysis and convergence of the suggested method are addressed in section 2. In Section 3, the numerical results obtained from applying new scheme to an illustrative example are presented. Finally, the conclusion is stated in Section 4.

Description of the Method

Let $h = 1/M$ be the step size in x direction, M and N be positive integers, and k explain the step size of time. Consider the following nodes $x_i = ih,\ i = 0,1,\dots,M,$ and $t_j = jk,\ \ j = 0,1,2,\dots,N.$ Moreover, let $t_{j+1/2} = (j+1/2)k$ and $u_{i,j} = u(x_i, t_j),$ where $0 \le i \le M,\ \ 0 \le j \le N$.

In the following, product trapezoidal method explained approximation of $I(u,t) = \int_0^t (t-s)^{-\alpha} u(s)\,ds$. This method was presented for $\alpha = 1/2$, by Tang [11]. To start, for any $u \in (C^1[0,1] \cap C^3(0,1])$ satisfying $u''(t) = O(t^{-\alpha})$ and $u'''(t) = O(t^{-1-\alpha})$, as $t \to 0^+$, $I(u,t)$ will be approximated numerically. Obviously,

$$I(u, t_{j+1/2}) = \frac{1}{2}\left[I(u,t_j) + I(u,t_{j+1}) \right] + O(k^2 t_j^{-1-\alpha}),\ \ j \ge 0. \tag{3}$$

Now, the product trapezoidal method is applied to approximate $I(u,t_j),\ 1 \le j \le N$. For $u(t_j - \theta)$ with $\theta \in [t_n, t_{n+1}],\ 0 \le n \le j-2$, the following can be written:

$$u(t_j - \theta) = \frac{t_{n+1} - \theta}{k} u(t_{j-n}) + \frac{\theta - t_n}{k} u(t_{j-n-1}) + E_{j1}, \tag{4}$$

for $\theta \in [t_{j-1}, t_j]$, the following will be given:

$$u(t_j - \theta) = \frac{t_n - \theta}{k} u(t_1) + \frac{\theta - t_{n-1}}{k} u(t_0) + O(k^{2-\alpha}). \tag{5}$$

The remainder term, E_{j1}, in (4) can be bounded by $O(k^2 t_{j-n-1}^{-\alpha}) = O(k^{2-\alpha})$. Using (4), (5) and transformation $\theta = t_j - s$, the following will be resulted:

$$\begin{aligned} I(u,t_j) &= \int_0^{t_j} \theta^{-\alpha} u(t_j - \theta)\,d\theta = \sum_{n=0}^{j-1} \int_{t_n}^{t_{n+1}} \theta^{-\alpha} u(t_j - \theta)\,d\theta \\ &= \sum_{n=0}^{j-1} \int_{t_n}^{t_{n+1}} \theta^{-\alpha} \left[\frac{t_{n+1} - \theta}{k} u(t_{j-n}) + \frac{\theta - t_n}{k} u(t_{j-n-1}) \right] d\theta + E_{j2}, \end{aligned} \tag{6}$$

where

$$\left| E_{j2} \right| \le \sum_{n=0}^{j-1} \int_{t_n}^{t_{n+1}} \theta^{-\alpha} O(k^{2-\alpha})\,d\theta = O(k^{2-\alpha}). \tag{7}$$

Using integration by parts in (6), results in:

$$\sum_{n=0}^{j-1}\int_{t_n}^{t_{n+1}}\theta^{-\alpha}\left[\frac{t_{n+1}-\theta}{k}u(t_{j-n})+\frac{\theta-t_n}{k}u(t_{j-n-1})\right]d\theta$$
$$=\frac{1}{1-\alpha}\sum_{n=0}^{j-1}[t_{n+1}^{1-\alpha}u(t_{j-n-1})-t_n^{1-\alpha}u(t_{j-n})]+\frac{1}{1-\alpha}\sum_{n=0}^{j-1}\int_{t_n}^{t_{n+1}}\theta^{1-\alpha}\frac{u(t_{j-n})-u(t_{j-n-1})}{k}d\theta,$$
$$=\lambda_j u(t_0)+\gamma_0 u(t_j)+\sum_{n=1}^{j-1}\gamma_n u(t_{j-n})+O(k^{2-\alpha}), \tag{8}$$

where

$$\lambda_j=\frac{1}{1-\alpha}[t_j^{1-\alpha}-\frac{1}{k}\int_{t_{j-1}}^{t_j}\theta^{1-\alpha}\,d\theta],$$
$$\gamma_0=-\frac{1}{1-\alpha}[t_0^{1-\alpha}-\frac{1}{k}\int_{t_0}^{t_1}\theta^{1-\alpha}\,d\theta], \tag{9}$$
$$\gamma_n=\frac{1}{k(1-\alpha)}[\int_{t_n}^{t_{n+1}}\theta^{1-\alpha}\,d\theta-\int_{t_{n-1}}^{t_n}\theta^{1-\alpha}\,d\theta].$$

The following lemma will be used in the derivation of the compact difference scheme.

Lemma1.1. Suppose that $u(x)\in C^6[x_{i-1},x_{i+1}]$, then a compact finite difference form of the following differential equation

$$\begin{cases}u''+\beta u'=f(x), & x\in(0,1),\\ u(0)=u(1)=0.\end{cases} \tag{10}$$

Is

$$(r_1+r_2)u_{i+1}-2r_1u_i+(r_1-r_2)u_{i-1}=(r_3+r_4)f_{i+1}+10r_3f_i+(r_3-r_4)f_{i-1} \tag{11}$$

Where

$$r_1=\frac{1}{h^2}+\frac{\beta^2}{12},\ r_2=\frac{\beta}{2h},\ r_3=\frac{1}{12},\ r_4=\frac{\beta h}{24}.$$

Proof: The well-known central difference approximation for the first and the second derivatives of u(x) are applied which result in the following discrete form of equation (10), at the x_i point:

$$\delta_x^2u_i+\beta\delta_xu_i-\tau_i=f_i, \tag{12}$$

where

$$\delta_x^2u_i=\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}+\frac{h^2}{12}u^{(4)},$$
$$\delta_xu_i=\frac{u_{i+1}-u_{i-1}}{2h}+\frac{h^2}{6}u''', \tag{13}$$
$$\tau_i=\frac{h^2}{12}(\frac{d^4u}{dx^4}+2\beta\frac{d^3u}{dx^3})+O(h^4).$$

In order to take a fourth-order compact difference scheme for Eq. (10), the third and fourth derivatives of $u(x)$, in (13) should be approximated [15]. Considering (10), one has:

$$(\frac{d^3u}{dx^3})_i=(\frac{df}{dx}-\beta\frac{d^2u}{dx^2})_i=(\delta_xf_i-\beta\delta_x^2u_i)+O(h^2),$$
$$(\frac{d^4u}{dx^4})_i=(\frac{d^2f}{dx^2}-\beta\frac{d^3u}{dx^3})_i=(\delta_x^2f_i-\beta\delta_xf_i+\beta^2\delta_x^2u_i)+O(h^2). \tag{14}$$

By substituting (14) into the third equation of (13), truncation error can be written as the following:

$$\tau_i=\frac{h^2}{12}\left[\delta_x^2f_i+\beta\delta_xf_i-\beta^2\delta_x^2u_i\right]+O(h^4). \tag{15}$$

By substituting (15) in (12), a fourth-order compact finite difference form of Eq. (10) can be obtained.

1-1. Implementation of the product trapezoidal method

Equation (1) can be introduced as follows

$$u_t=\mu(u_{xx}+\beta u_x)+I(u_{xx}+\beta u_x,t). \tag{16}$$

According to lemma1.1, one has:

$$(\delta_x^2+\beta\delta_x)u_{i,j}=(r_1+r_2)u_{i+1,j}-2r_1u_{i,j}+(r_1-r_2)u_{i-1,j}. \tag{17}$$

By using (17), an application of the standard Crank-Nicolson method for (16) gives:

$$\frac{1}{12k}\Big[[(1+r_5)u_{i+1,j+1}+10u_{i,j+1}+(1-r_5)u_{i-1,j+1}]-[(1+r_5)u_{i+1,j}+10u_{i,j}+(1-r_5)u_{i-1,j}\Big]=$$
$$\mu(\delta_x^2+\beta\delta_x)\overline{u}_{i,j}+\frac{\lambda_j+\lambda_{j+1}+\gamma_{j+1}}{2}(\delta_x^2+\beta\delta_x)u_{i,0}+\frac{1}{2}\sum_{n=0}^{j}\gamma_n(\delta_x^2+\beta\delta_x)\overline{u}_{i,j-n}+E, \tag{18}$$

where λ_j,λ_{j+1} and γ_{j+1} are presented in (9), $r_5=\beta h/2$, and $\overline{u}_{i,j}=u_{i,j}+u_{i,j+1}$, $\overline{u}_{i,j-n}=u_{i,j-n}+u_{i,j-n+1}$.

There exists a constant c, such that the remainder term (E) in (18) can be predicted as follows:

$$|E|\le c(k^2+h^4+k^{2-\alpha}+k^2t_j^{-1-\alpha}).$$

Considering initial and boundary conditions, (2) can be written as the following:

$$u_{0,j} = u_{M,j} = 0,\ \ 0 \le j \le N,$$
$$u_{i,0} = u(x_i, 0),\ \ 1 \le i \le M-1. \tag{19}$$

In what follows the operator Δ, will be used,

$$\Delta u_{i,j} = \frac{1}{12k}\Big[[(1+r_5)u_{i+1,j+1} + 10u_{i,j+1} + (1-r_5)u_{i-1,j+1}] - [(1+r_5)u_{i+1,j} + 10u_{i,j} + (1-r_5)u_{i-1,j}]\Big] = \frac{1}{12}[(1+r_5)\delta_t u_{i+1,j} + 10\delta_t u_{i,j} + (1-r_5)\delta_t u_{i-1,j}]. \tag{20}$$

Eliminating E in (18) and replacing $u_{i,j}$ by $U_{i,j}$, the proposed compact scheme is constructed as follows:

$$\Delta U_{i,j} = \mu(\delta_x^2 + \beta\delta_x)\bar{U}_{i,j} + \tilde{\lambda}_j(\delta_x^2 + \beta\delta_x)U_{i,0} + \sum_{n=0}^{j}\gamma_n(\delta_x^2 + \beta\delta_x)\bar{U}_{i,j-n}, \tag{21}$$

where $\tilde{\lambda}_j = \dfrac{\lambda_j + \lambda_{j+1} + \gamma_{j+1}}{2}$.

Furthermore,

$$U_{0,j} = U_{M,j} = 0,\ \ j = 0,1,...,N,$$
$$U_{i,0} = U(x_i, 0),\ \ i = 0,1,...,M. \tag{22}$$

Analysis of the compact difference scheme

In this section, it will be shown that the proposed method is convergent with the order $O(h^4, k^{2-\alpha})$. In the special case of (1), when $\alpha = 1/2$ and $\beta = 0$, Luo et.al [13] proved that convergence order is $O(h^4, k^{3/2})$. Suppose U_h be the space of grid functions as follows:

$$U_h = \{U \,|\, U = (U_0, U_1, ..., U_{M-1}, U_M),\ U_0 = U_M = 0\}.$$

Defining the grid function $U_{i,j} = u(x_i, t_j),\ 0 \le i \le M, 0 \le j \le N$, for any two grid functions $U, W \in U_h$, one gets the followings;

$$\delta_t U_{i,j} = \frac{1}{k}(U_{i,j+1} - U_{i,j}),\quad \delta_x U_i = \frac{1}{2h}(U_{i+1} - U_{i-1}),\quad \delta_x^2 U_i = \frac{1}{h^2}(U_{i+1} - 2U_i + U_{i-1}),$$
$$\bar{U}_{i,j} = \frac{1}{2}(U_{i,j+1} + U_{i,j}),$$
$$(UW)_i = U_i W_i,\quad \|U\|_\infty = \max_{1\le i\le M-1}|U_i|,\quad \langle U, W\rangle = h\sum_{i=1}^{M-1} U_i W_i,\quad \|U\|^2 = \langle U, U\rangle. \tag{23}$$

Lemma2.1.

I) Suppose that $U, W \in U_h$, then

$$\langle \delta_x^2 U, W\rangle = -h\sum_{i=0}^{M-1}(\delta_x U)(\delta_x W). \tag{24}$$

II) If $U_{.,k}, U_{.,l} \in U_h$, then

$$\left|\langle(\delta_x^2 + \beta\delta_x)U_{.,k}, U_{.,l}\rangle\right| \le \frac{4+\beta}{h^2}\|U_{.,k}\|\|U_{.,l}\|.$$

Proof: For proving I, see [7].

(II): When $N \ge 1$, one has:

$$\left|\langle(\delta_x^2 + \beta\delta_x)U_{.,k}, U_{.,l}\rangle\right| \le \left|\langle\delta_x^2 U_{.,k}, U_{.,l}\rangle\right| + \beta\left|\langle\delta_x U_{.,k}, U_{.,l}\rangle\right| = \left|\sum_{i=1}^{M-1} h\delta_x^2 U_{i,k}U_{i,l}\right| + \beta\left|\sum_{i=1}^{M-1} h\delta_x U_{i,k}U_{i,l}\right|, \tag{25}$$

each term in (25) can be estimated. First, it can be written as the following:

$$\left(\left|\sum_{i=1}^{M-1} hU_{i+1,k}U_{i,l}\right|\right)^2 \le \left(\sum_{i=1}^{M-1} h|U_{i+1,k}||U_{i,l}|\right)^2 \le \left(\sum_{i=1}^{M-1} h(U_{i+1,k})^2\right)\left(\sum_{i=1}^{M-1} h(U_{i,l})^2\right) \le \|U_{.,k}\|^2\|U_{.,l}\|^2, \tag{26}$$

by using Cauchy-Schwarz inequality, one gets:

$$\|\delta_x U_{.,k}\|^2 = h\sum_{i=0}^{M-1}|\delta_x U_{i,k}|^2 = h\sum_{i=1}^{M-1}\left|\frac{U_{i+1,k} - U_{i-1,k}}{2h}\right|^2 \le \frac{1}{4h}\left[\sum_{i=0}^{M-1}|U_{i+1,k}|^2 + 2\sum_{i=1}^{M-1}|U_{i+1,k}U_{i-1,k}|^2 + \sum_{i=0}^{M-1}|U_{i-1,k}|^2\right] \le \frac{1}{h^2}\|U_{.,k}\|^2. \tag{27}$$

Consequently, inequality (25) can be written as the following:

$$\left|\langle(\delta_x^2 + \beta\delta_x)U_{.,k}, U_{.,l}\rangle\right| \le \left|\sum_{i=1}^{M-1} h\delta_x^2 U_{i,k}U_{i,l}\right| + \beta\left|\sum_{i=1}^{M-1} h\delta_x U_{i,k}U_{i,l}\right| = \frac{1}{h}\left|\sum_{i=1}^{M-1}(U_{i+1,k} - 2U_{i,k} + U_{i-1,k})U_{i,l}\right| + \frac{\beta}{2}\left|\sum_{i=1}^{M-1}(U_{i+1,k} - U_{i-1,k})U_{i,l}\right|,$$

Using (26) and (27), yields to:

$$\left|\langle(\delta_x^2 + \beta\delta_x)U_{.,k}, U_{.,l}\rangle\right| \le \frac{4+\beta}{h^2}\|U_{.,k}\|\|U_{.,l}\|. \tag{28}$$

Which completes the proof.

Lemma2.2. Let $U_{.,j} = (U_{1,j}, ..., U_{M-1,j})$, then

$$k\sum_{j=0}^{N-1}\left\langle \Delta U_{.,j},\bar{U}_{.,j}\right\rangle \geq \frac{11}{24}\left\|U_{.,N}\right\|^2-\frac{1}{2}\left\|U_{.,0}\right\|^2.$$

Proof: By using (20), the general term of this sequence can be written as the following:

$$\begin{aligned}\left\langle \Delta U_{.,j},\bar{U}_{.,j}\right\rangle &= \frac{h}{12}\sum_{i=1}^{M-1}([(1+r_5)\delta_t U_{i+1,j}+10\delta_t U_{i,j}+(1-r_5)\delta_t U_{i-1,j}])(\bar{U}_{i,j})\\ &= h\sum_{i=1}^{M-1}(\delta_t U_{i,j}+\frac{h^2}{12}\delta_x^2\delta_t U_{i,j}+\frac{r_5 h}{6}\delta_x\delta_t U_{i,j})(\bar{U}_{i,j})\\ &=\left\langle \delta_t U_{.,j},\bar{U}_{.,j}\right\rangle+\frac{h^2}{12}\left\langle \delta_x^2\delta_t U_{.,j},\bar{U}_{.,j}\right\rangle+\frac{r_5 h}{6}\left\langle \delta_x\delta_t U_{.,j},\bar{U}_{.,j}\right\rangle\\ &=\frac{1}{2k}(\left\|U_{.,j+1}\right\|^2-\left\|U_{.,j}\right\|^2)-\frac{h^2}{24k}(\left\|\delta_x U_{.,j+1}\right\|^2-\left\|\delta_x U_{.,j}\right\|^2)-\frac{r_5 h}{6}\left\|\delta_x U_{.,j}\right\|^2\left\|\delta_t \bar{U}_{.,j}\right\|^2\\ &\geq \frac{1}{2k}\left[(\left\|U_{.,j+1}\right\|^2-\frac{h^2}{12}\left\|\delta_x U_{.,j+1}\right\|^2)-(\left\|U_{.,j}\right\|^2-\frac{h^2}{12}\left\|\delta_x U_{.,j}\right\|^2)\right],\end{aligned} \tag{29}$$

and by considering (27)

$$\left\|U_{.,j}\right\|^2 \geq \left\|U_{.,j}\right\|^2-\frac{h^2}{12}\left\|\delta_x U_{.,j}\right\|^2 \geq \left\|U_{.,j}\right\|^2-\frac{1}{12}\left\|U_{.,j}\right\|^2 \geq \frac{11}{12}\left\|U_{.,j}\right\|^2. \tag{30}$$

Using (29) and (30), leads to:

$$\begin{aligned}k\sum_{j=0}^{N-1}\left\langle \Delta U_{.,j},\bar{U}_{.,j}\right\rangle &\geq \frac{1}{2}\left[(\left\|U_{.,N}\right\|^2-\frac{h^2}{12k}\left\|\delta_x U_{.,N}\right\|^2)-(\left\|U_{.,0}\right\|^2-\frac{h^2}{12k}\left\|\delta_x U_{.,0}\right\|^2)\right]\\ &\geq \frac{11}{24}\left\|U_{.,N}\right\|^2-\frac{1}{2}\left\|U_{.,0}\right\|^2.\end{aligned} \tag{31}$$

Now, the stability and the convergence of the proposed approach will be proved.

2.1. Stability

The stability of the scheme by means of the energy method should be established as the following:

Theorem2.1. Let $U_{.,j}=(U_{1,j},...,U_{M-1,j})$ for $i=1,...,M-1, j=1,...,N$ be the solution of the following equation

$$\Delta U_{i,j}=\mu(\delta_x^2+\beta\delta_x)\bar{U}_{i,j}+\tilde{\lambda}_j(\delta_x^2+\beta\delta_x)U_{i,0}+\sum_{n=0}^{j}\gamma_n(\delta_x^2+\beta\delta_x)\bar{U}_{i,j-n}, \tag{32}$$

with initial and boundary conditions (22). Then for $N\geq 1$,

$$\left\|U_{.,N}\right\| \leq \frac{Ck^{2-\alpha}T^\alpha}{11h^2}\left\|U_{.,0}\right\|+\frac{12}{11}\left\|U_{.,0}\right\|. \tag{33}$$

Proof: Multiplying both sides of (32) by $h\bar{U}_{i,j}$ and summing i from 1 up to $M-1$, and j from 0 up to N, the following can be written:

$$\begin{aligned}\sum_{j=0}^{N-1}\left\langle \Delta U_{.,j},\bar{U}_{.,j}\right\rangle &= \mu\sum_{j=0}^{N-1}\left\langle (\delta_x^2+\beta\delta_x)\bar{U}_{.,j},\bar{U}_{.,j}\right\rangle\\ &+\sum_{j=0}^{N-1}\left\langle \tilde{\lambda}_j(\delta_x^2+\beta\delta_x)U_{.,0},\bar{U}_{.,j}\right\rangle+\sum_{j=0}^{N-1}\sum_{n=0}^{j}\gamma_n\left\langle (\delta_x^2+\beta\delta_x)\bar{U}_{.,j-n},\bar{U}_{.,j}\right\rangle.\end{aligned} \tag{34}$$

The first term in the right side of equation (34), when using (26), will be estimated as follows:

$$\begin{aligned}k\mu\left\langle (\delta_x^2+\beta\delta_x)\bar{U}_{.,j},\bar{U}_{.,j}\right\rangle &= k\mu h\sum_{i=1}^{M-1}((\delta_x^2+\beta\delta_x)\bar{U}_{i,j})(\bar{U}_{i,j})\\ &= k\mu h\sum_{i=1}^{M-1}(\delta_x^2\bar{U}_{i,j})(\bar{U}_{i,j})+k\mu h\sum_{i=1}^{M-1}(\beta\delta_x\bar{U}_{i,j})(\bar{U}_{i,j})\\ &= -k\mu h\sum_{i=1}^{M-1}(\delta_x\bar{U}_{i,j})(\delta_x\bar{U}_{i,j})+k\mu\beta h\sum_{i=1}^{M-1}(\delta_x\bar{U}_{i,j})(\bar{U}_{i,j})\\ &= -k\mu\left\|\bar{U}_{.,j}\right\|^2-\frac{k\mu\beta}{h}\left\|\bar{U}_{.,j}\right\|^2=-k\mu(1+\frac{\beta}{h})\left\|\bar{U}_{.,j}\right\|^2\leq 0.\end{aligned} \tag{35}$$

In the third equality of (35), Lemma 2.1(1) is used.

For estimation of the second term on the right hand side of equality (34), using Cauchy–Schwarz inequality and (28), result in:

$$\begin{aligned}k\sum_{j=0}^{N-1}\left\langle \tilde{\lambda}_j(\delta_x^2+\beta\delta_x)U_{.,0},\bar{U}_{.,j}\right\rangle &= kh\sum_{j=0}^{N-1}\sum_{i=1}^{M-1}\tilde{\lambda}_j((\delta_x^2+\beta\delta_x)U_{i,0})\bar{U}_{i,j}\\ &\leq k\,|\sum_{j=0}^{N-1}\tilde{\lambda}_j\,|\left\langle (\delta_x^2+\beta\delta_x)U_{.,0},\bar{U}_{.,j}\right\rangle\\ &\leq \frac{4+\beta}{h^2}k\,|\sum_{j=0}^{N-1}\tilde{\lambda}_j\,|\left\|U_{.,0}\right\|\left\|\bar{U}_{.,j}\right\|.\end{aligned} \tag{36}$$

According to (9), and using the mean value theorem of integrals, the sum of λ_j can be written as the following:

$$\begin{aligned}k\sum_{j=0}^{N-1}\lambda_j &= k\sum_{j=0}^{N-1}\frac{1}{1-\alpha}[t_j^{1-\alpha}-\frac{1}{k}\int_{t_{j-1}}^{t_j}\theta^{1-\alpha}d\theta]=\frac{k}{1-\alpha}\sum_{j=0}^{N-1}(t_j^{1-\alpha}-\eta_1^{1-\alpha})\leq k\sum_{j=0}^{N-1}kt_j^{\alpha-1}\\ &= k^2\frac{k}{k^\alpha}\left[\frac{1}{(k)^{1-\alpha}}+\frac{1}{(2k)^{1-\alpha}}+\cdots+\frac{1}{(mk)^{1-\alpha}}\right]\leq k^{2-\alpha}\int_0^T\frac{1}{x^{1-\alpha}}dx\leq\frac{1}{\alpha}k^{2-\alpha}T^\alpha,\end{aligned} \tag{37}$$

where $\eta_1\in(t_j,t_{j+1})$, and

$$k\sum_{j=0}^{N-1}\gamma_{j+1}=k\sum_{j=0}^{N-1}\frac{1}{k(1-\alpha)}[\int_{t_j}^{t_{j+1}}\theta^{1-\alpha}d\theta-\int_{t_{j-1}}^{t_j}\theta^{1-\alpha}d\theta]=\frac{k}{1-\alpha}\sum_{j=0}^{N-1}(\eta_2^{1-\alpha}-\eta_3^{1-\alpha})$$
$$\le k\sum_{j=0}^{N-1}(t_{j+2}^{\alpha-1}-t_j^{\alpha-1})\le 2k^2\sum_{j=0}^{N-1}t_j^{\alpha-1}\le 2k^{2-\alpha}\int_0^T\frac{1}{x^{1-\alpha}}dx\le Ck^{2-\alpha}T^{\alpha}, \quad (38)$$

where $\eta_2\in(t_j,t_{j+1})$, and $\eta_3\in(t_{j-1},t_j)$.

The last term on the right hand side of equality (34), will be estimated as follows:

$$k\sum_{j=0}^{N-1}\sum_{n=0}^{j}\gamma_n\left\langle(\delta_x^2+\beta\delta_x)\bar{U}_{.,j-n},\bar{U}_{.,j}\right\rangle\le -kh\sum_{j=0}^{N-1}|\sum_{n=0}^{j}\gamma_n|((\delta_x^2+\beta\delta_x)\bar{U}_{.,j-n})(\bar{U}_{.,j})$$
$$\le -C\frac{4+\beta}{h^2}Nk^{2-\alpha}T^{\alpha}\|\bar{U}_{.,j-n}\|\|\bar{U}_{.,j}\|\le 0. \quad (39)$$

Since $\tilde{\lambda}_j=\frac{1}{2}(\lambda_j+\lambda_{j+1}+\gamma_{j+1})$ and substituting (31) and (35)–(39) into (34), this inequality can be obtained as follows:

$$\frac{11}{24}\|U_{.,N}\|^2-\frac{1}{2}\|U_{.,0}\|^2\le k\sum_{j=0}^{N-1}\left\langle\Delta U_{.,j},\bar{U}_{.,j}\right\rangle\le k\sum_{j=0}^{N-1}\left\langle\tilde{\lambda}_j(\delta_x^2+\beta\delta_x)U_{.,0},\bar{U}_{.,j}\right\rangle$$
$$\le\frac{4+\beta}{h^2}|\sum_{j=0}^{N-1}\tilde{\lambda}_j|\|U_{.,0}\|\|\bar{U}_{.,j}\|$$
$$\le\frac{4+\beta}{2h^2}(\frac{1}{\alpha}k^{2-\alpha}T^{\alpha}+\frac{1}{\alpha}k^{2-\alpha}T^{\alpha}+Ck^{2-\alpha}T^{\alpha})\|U_{.,0}\|\|\bar{U}_{.,j}\|,$$

so

$$\|U_{.,N}\|^2\le\frac{12(4+\beta)}{11h^2}(\frac{2}{\alpha}k^{2-\alpha}T^{\alpha}+Ck^{2-\alpha}T^{\alpha})\|U_{.,0}\|\|\bar{U}_{.,j}\|+\frac{12}{11}\|U_{.,0}\|^2,$$

choosing J, so that $\|U_{.,J}\|=\max_{0\le j\le N}\|U_{.,j}\|$, results in;

$$\|U_{.,J}\|^2\le\frac{12(4+\beta)}{11h^2}(\frac{2}{\alpha}k^{2-\alpha}T^{\alpha}+Ck^{2-\alpha}T^{\alpha})\|U_{.,0}\|\|\bar{U}_{.,J}\|+\frac{12}{11}\|U_{.,0}\|^2$$
$$\le\frac{Ck^{2-\alpha}T^{\alpha}}{11h^2}\|U_{.,0}\|\|U_{.,J}\|+\frac{12}{11}\|U_{.,0}\|\|U_{.,J}\|,$$

Therefore, for $N\ge 1$, above inequality can be written as the following:

$$\|U_{.,N}\|\le\|U_{.,J}\|\le\frac{Ck^{2-\alpha}T^{\alpha}}{11h^2}\|U_{.,0}\|+\frac{12}{11}\|U_{.,0}\|,$$

which completes the proof.

2.2. Convergence

The convergence of the numerical method (21), with initial and boundary conditions (22), is proved similar to Theorem 2.1. Denote

$$e_{i,j}=u_{i,j}-U_{i,j},\quad 0\le i\le M,\ 0\le j\le N.$$

Theorem2.2. Assume that $\{u_{i,j}:\ 0\le i\le M,\ 0\le j\le N\}$ is a solution of the problem (1), subjected to initial and boundary conditions (2). Let $(U_{.,0},U_{.,1},...,U_{.,N})$ be the solution of compact difference scheme (21) with initial and boundary conditions (22). When h and k tend to zero independently, leads to:

$$\max_{1\le j\le N}\|e_{.,j}\|=O(h^4,k^{2-\alpha}). \quad (40)$$

Proof: Subtracting (18) and (19) from (21) and (22), respectively, the error operator can be obtained as the following:

$$\Delta e_{i,j}=\mu(\delta_x^2+\beta\delta_x)\bar{e}_{i,j}+\sum_{n=0}^{j}\gamma_n(\delta_x^2+\beta\delta_x)\bar{e}_{i,j-n}+E,\quad 1\le i\le M-1, 1\le j\le N-1, \quad (41)$$

and
$$e_{0,j}=e_{M,j}=0,\quad j=0,1,...,N.$$
$$e_{i,0}=0,\quad i=1,...,M-1.$$

Multiplying (41) by $kh\bar{e}_{i,j}$ and summing on i from 1 up to $M-1$, and j from 0 to N, can be obtained as follows:

$$k\sum_{j=0}^{N-1}\left\langle\Delta e_{.,j},\bar{e}_{.,j}\right\rangle=k\mu\sum_{j=0}^{N-1}\left\langle(\delta_x^2+\beta\delta_x)\bar{e}_{.,j},\bar{e}_{.,j}\right\rangle$$
$$+k\sum_{j=0}^{N-1}\sum_{n=0}^{j}\gamma_n\left\langle(\delta_x^2+\beta\delta_x)\bar{e}_{.,j-n},\bar{e}_{.,j}\right\rangle+k\sum_{j=0}^{N-1}\left\langle E,\bar{e}_{.,j}\right\rangle. \quad (42)$$

As in the proof of Theorem 2.1, choosing $\|e_{.,J}\|=\max_{0\le j\le N}\|e_{.,j}\|$ results in;

$$\frac{11}{24}\|e_{.,J}\|^2-\frac{1}{2}\|e_{.,0}\|^2\le k\sum_{j=0}^{N-1}\left\langle E,\bar{e}_{.,j}\right\rangle\le Ck\sum_{j=0}^{N-1}\|E\|_{\infty}.\|e_{.,J}\|\le[C\sum_{j=0}^{N-1}(k^{3-\alpha}+kh^4+k^3t_{j+1}^{-1-\alpha})]\|e_{.,J}\|$$
$$\le[Ck^{2-\alpha}\sum_{j=0}^{N-1}(j+1)^{-1-\alpha}+Ch^4+Ck^{2-\alpha}+O(h^4+k^{2-\alpha})]\|e_{.,J}\|, \quad (43)$$

therefore, $\|e_{.,J}\|^2\le 3C(h^4+k^{2-\alpha})\|e_{.,J}\|$.

Hence, the convergence order (40) is obtained.

Results

In order to illustrate our schemes, the following examples are computed.

Example 1.

The effectiveness of scheme (21) is demonstrated by the following example. Consider the partial integro-differential equation

$$u_t = u_{xx} + \int_0^t (s-t)^{-\frac{1}{2}} u_{xx} ds,$$
$$u(0,t) = u(1,t) = 0,\ \ 0 \le t \le T.$$
$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1.$$

Let $\beta = 0$, $\mu = 1$, $\alpha = 1/2$, and $T = 0.5$. Take $h = 1/M$, $k = 1/N$ and $M = 10$. The numerical solution of $u(x,t)$ corresponding to $M \times N = 10 \times 640$ is used as the exact solution. The results are presented in Table 1 with different step-sizes.

The numerical results from Table 1 reflect that the convergence rate in time is $3/2$. Our results are similar to the numerical solution reported in [14], but our method can be applied to any α. Comparing the results in Table 1 with those of table 4, in [14], the results obtained in this paper are more accurate. For example, consider $N = 20$, the error in [14], is $1.00346e-003$, but the error in this paper is $3.60254e-004$.

Example 2.

Consider Eq. (1) when $\alpha = 1/2$ and $\mu = \beta = 0$, with the following exact solution $u(x,t) = \Psi\sqrt{\pi^5 t^3}\sin(\pi x)$.

where Ψ, denotes the entire function

$$\Psi(z) = \sum_{i=0}^{\infty} (-1)^i \Gamma(\frac{3}{2}i + 1)^{-1} z^i \,. \qquad (44)$$

The initial and boundary conditions can be obtained from the exact solution. This test problem is used in [14]. The authors of [14] proposed a scheme which has fourth-order accuracy in space and $\frac{3}{2}$ in time. The accuracy of our method is tested by solving this problem with several values of steps size and presents rates of convergence in time for $T = 0.5$. Our results are similar to the numerical solution in [14] because we used the same method. The numerical results are presented in Table 2.

Discussion

In this article, a compact difference scheme, for a partial integro-differential equation with a weakly singular kernel for any $0 < \alpha < 1$, is constructed. The stability and L_2 norm convergence is proved by energy method. In this study, Crank–Nicolson time-stepping is

Table 1. Maximum error for example 1

N	Error in [12]	Error in [14]	Error	Rate
10	$2.49e-002$	$2.87067e-003$	$1.01914e-003$	--
20	$8.66e-003$	$1.00346e-003$	$3.60254e-004$	1.50026
40	$3.05e-003$	$3.52049e-004$	$1.25719e-004$	1.51881
80	--	$1.21344e-004$	$4.35397e-005$	1.52979

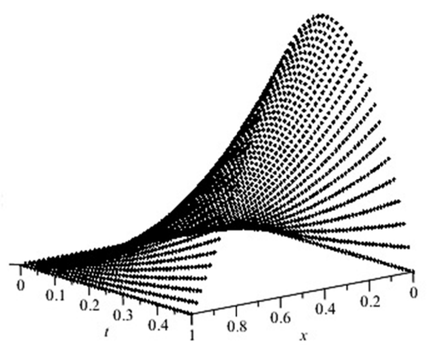

Figure 1. Computational solution of problem 2

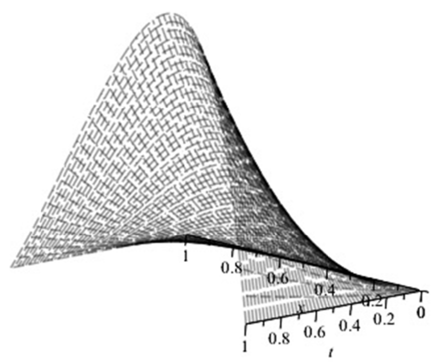

Figure 2. Computational solution of problem 2

Table 2. Maximum error for example 2

N	Error in [14]	Rate in [14]	Error	Rate
20	8.93117e−002	--	9.05317e−003	--
40	3.21568e−002	1.47372	3.19954e−003	1.50055
80	1.15715e−002	1.47455	1.13096e−003	1.50031
160	4.16970e−003	1.47256	3.99814e−004	1.50014
320	1.50417e−003	1.47098	1.41360e−004	1.49995
640	5.46742e−004	1.46004	5.00614e−005	1.49760

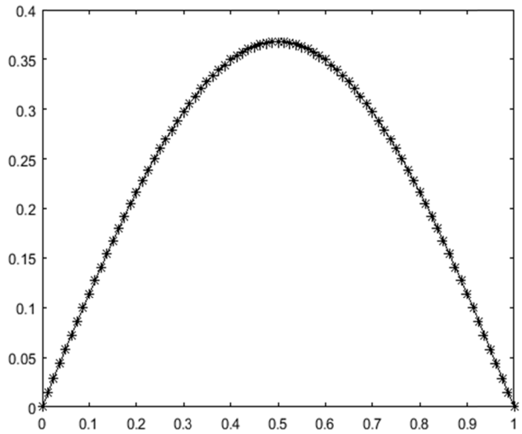

Figure 3. Numerical approximations are compared with exact solution (44) of problem 2 for $N = 80$.

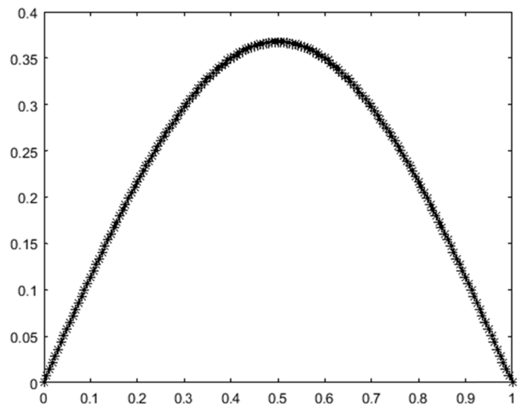

Figure 4. Numerical approximations are compared with exact solution (44) of problem 2 for $N = 160$.

used to approximate the differential term and a product trapezoidal method is used to approximate the integral term. The convergence order is $2-\alpha,$ in time and 4 in space. The method was tested against exact reference solutions of two different examples, where $\beta = 0$ and $\mu = 1$, and $\beta, \mu = 0$, both examples supported our theoretical results. What's more by increasing N, the rates in time will increase at a steady rate. In addition, numerical results presented in the present study are more accurate than those reported in [12, 14]. It is worth mentioning that our computations are performed by Matlab.

Acknowledgment

We would like to thank both referees for their valuable comments and helpful suggestions which have

improved the paper.

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