Non-Relativistic Limit of Neutron Beta-Decay Cross-Section in the Presence of Strong Magnetic Field

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Abstract

One of the most important reactions of the URCA that lead to the cooling of a neutron star, is neutron beta-decay ($n \rightarrow p^+ + e^- + \nu$). In this research, the energy spectra and wave functions of massive fermions taking into account the Anomalous Magnetic Moment (AMM) in the presence of a strong changed magnetic field are calculated. For this purpose, the Dirac-Pauli equation for charged and neutral fermions is solved by Perturbation and Frobenius series method, respectively. The results of the Frobenius series method are in good agreement with the results of Nikiforov-Uvarov method (NU). In continuous, using the calculated wave functions, the general relation of neutron decay cross-section in the non-relativistic limit has been obtained. This relation has been derived by the four-fermion Lagrangian within the framework of the standard model of weak interactions. These calculations from the perspective of nuclear astrophysics can be important.

Keywords: Dirac-Pauli equation; Anomalous Magnetic Moment; Four-Fermion Lagrangian.

Introduction

Study of weak processes in the presence of strong magnetic fields in neutron stars and Radio-pulsars are very important in view of astrophysical applications. The neutron stars are super-dense objects with density about of $2.8 \times 10^{14} gr/cm^3$ [1]. The strength of the surface magnetic field in these objects is in the range of $10^{12} \sim 10^{14} G$ [2, 3]. Also, the intensity of the magnetic fields can be $10^{14} \sim 10^{15} G$ for some magnetars [4, 5]. For more details about the properties of magnetars, refer to papers by Gao et al. [6] and Mereghetti et al. [7]. They showed that magnetars are neutron stars in which a strong magnetic field is the main energy source.

The neutron stars are the best laboratory for studying dense matter physics. These objects are caused by supernova explosions with an internal temperature about of $10^{11} \sim 10^{12} K$. The processes of cooling in the neutron stars are due to emitting neutrinos. During these processes temperature decreases to $T \approx 10^9 K$ [8, 9]. The weak interactions of nucleons cause the beta-decay and the emission of neutrinos from neutron stars (the URCA process) [10, 11].

The strong magnetic fields will strongly influence on the weak interactions. The details of weak interactions related to the URCA process are presented in Ref. [12]. Even relatively weak fields can play strong role in various astrophysical problems [13].

Up to now, it is widely recognized that strong magnetic fields can be a significant factor related to

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Materials and Methods

For a description of particles with half spin in relativistic limit, we use the Dirac equation. In the non-relativistic limit the Dirac equation convert to the Dirac–Pauli or Pauli equation. In this section we give a brief discussion of the Dirac-Pauli equation with AMM in external field and cross-section of beta-decay.

**Dirac-Pauli equation with AMM in external field**

In the relativistic quantum theory, moving of a charged particle with half spin in the external electromagnetic field describes by the Dirac equation [27, 28]. In the non-relativistic limit to obtain the energy spectrum and wave function of moving fermions in an external magnetic field, it is necessary to convert the Dirac equation to the Pauli equation. The Dirac equation for fermions with AMM in external magnetic field can be written as following [28]:

\[
\frac{i\hbar}{\gamma} \psi(r, t) = \left[ \gamma \cdot \mathbf{P} + \gamma_0 m - \mathbf{\mu} \psi \left( \mathbf{\gamma} \cdot \mathbf{B} \right) \right] \psi(r, t),
\]

(1)

Parameters in this equation are defined as follows:

\[
\rho = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \rho_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \mathbf{P} = \gamma_0 \mathbf{\sigma} - \mathbf{A}, \quad \mathbf{E} = \mathbf{A} \times \mathbf{B}, \quad \mathbf{B} = \mathbf{\sigma} \cdot \mathbf{J}.
\]

(2)

Where \( \mathbf{\sigma} \) - are Pauli matrices, \( \mathbf{A} \) - is the vector potential of magnetic field \( \mathbf{I} \) and \( \mathbf{0} \) - are the \( 2 \times 2 \) matrixes, \( m \) - is fermion mass and \( \mathbf{\mu} \) - is AMM of particle. One can easily obtain the Pauli equation with contribution of AMM of fermions, so we have:

\[
\frac{i\hbar}{\gamma} \psi(r, t) = \left[ \gamma \cdot \mathbf{P} + \gamma_0 m - \mathbf{\mu} \psi \left( \mathbf{\gamma} \cdot \mathbf{B} \right) \right] \psi(r, t).
\]

(3)

The stationary state of bispinor \( \psi(r, t) \) is defined as

\[
\psi(r, t) = \exp(-i\mathbf{E}t) \begin{pmatrix} \psi_1(r) \\ \psi_2(r) \end{pmatrix}.
\]

(4)

Where \( \mathbf{E} \) - is the energy of fermion in the non-relativistic limit. We are interested in solving the Eq. (3) for neutral and charged fermions in the presence of an external magnetic field. We introduced the magnetic field with cylindrical symmetry:

\[
\mathbf{B}(r) = \begin{pmatrix} 0, 0, a r/b + b \end{pmatrix}.
\]

(5)

In the Eq. (5), \( a \) and \( b \) are constant parameters. The vector potential for such magnetic field becomes:

\[
\mathbf{A}_r = A_z = 0, \quad A_\phi = \frac{br}{2} + a.
\]

(6)
There exist many gauges which produce the magnetic field, but for convenience we introduced the gauge of mentioned in the Eq. (6).

**Cross-section for beta-decay in an external field**

In order to calculating the beta-decay of cross-section within the framework of the standard model of weak interactions we use the four-fermion Lagrangian as follow [21, 26]:

\[
1 = \frac{G_F}{\sqrt{2}} \left[ \bar{\psi} \gamma_\mu (1 - \gamma_5) \psi \right] \left[ \bar{\psi'} \gamma^\nu (1 - \gamma_5) \psi' \right] \gamma_i = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \text{ for } i = 1, 2, 3 .
\] (7)

In above \( G_F \) is the Fermi coupled constant and \( \alpha = 1.26 \) is the ratio of the axial and vector constants. The total cross section of the process can be written as:

\[
\sigma = \frac{\alpha^2}{T} \sum_{phase \ space} |M|^2 \tag{8}
\]

In the Eq. (8) summation is performed over the phase space of the final particles and \( M \) is the matrix element of the process. The matrix element is related to the four-fermion Lagrangian can be written as:

\[
M = \int d^4x \frac{G_F}{\sqrt{2}} \left[ \bar{\psi} \gamma_\mu (1 - \gamma_5) \psi \right] \left[ \bar{\psi'} \gamma^\nu (1 - \gamma_5) \psi' \right] d^4x \tag{9}
\]

We account for the influence of the background magnetic field in the Eq. (9). In this study the massive fermions have been considered in the non-relativistic limit. Whereas the neutrino is massless, so neutrino cannot be non-relativistic. The neutrinos have not interaction with magnetic field, furthermore they have not AMM, therefore the neutrino wave function in the presence of constant magnetic field the same as changed magnetic field. According to the matter, we use the neutrino relativistic wave function in our calculations which is denoted by

\[
\psi_\nu \left( \vec{r}, t \right) = \frac{1}{2L^2} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \exp \left( i \left( \chi_0 t - \vec{\chi} \cdot \vec{r} \right) \right) \tag{10}
\]

\[
f_1 = -e^{i \theta} \left( 1 - \cos \theta \right)^{1/2} , \quad f_2 = (1 + \cos \theta \theta_j)^{1/2} \tag{11}
\]

Here \( \chi_0 \cdot \vec{\chi} \) are the neutrino energy and momentum respectively. We neglect effects of the neutrino mass, so \( \chi_0 = \vec{\chi} \). The neutrino polar and azimuthally angles are denoted by \( \phi_\nu \) and \( \theta_\nu \) respectively [1, 21].

**Results**

To obtain the wave functions and the energy spectra of the massive fermions (electron, proton and neutron), we will solve the Dirac-Pauli equation, then we will use the wave functions for calculating the cross-section of neutron decay. This section contains three subsections which are: neutral fermion, charged fermion and the general relation of cross-section in the non-relativistic limit.

**Neutral fermion**

Substituting the Eq. (5) and Eq. (6) into Eq. (3) for neutral fermion we have

\[
\psi_{1,2} \left( \vec{r} \right) = \frac{1}{2L^2} \sum_{l=0,1,2L} \left( r - eA \right)^2 \frac{\mu + e}{2m} \left( \frac{\alpha}{r} + \beta \right) \psi_{1,2} \left( \vec{r} \right) . \tag{12}
\]

Solutions of this equation in cylindrical coordinates are:

\[
\psi_{1,2} \left( \vec{r} \right) = \frac{1}{2L^2} \sum_{l=0,1,2L} |J_l (k_j r)| \cdot \exp \left( i (l \phi + k_j z) \right) \cdot \left( C_{ln} \right) \tag{13}
\]

In this equation, radial function in cylindrical coordinates, \( f \left( r \right) \), is calculated using the Frobenious series method as follows:

\[
f \left( r \right) = e^{-\sqrt{A} r} F_1 \left( \frac{n + l + 1}{2} \right) , \tag{14}
\]

\[
A = k_j^2 - \left[ 2m_n \left( \mu_n + \mu_b + \varepsilon_n \right) \right] , \quad x = 2\sqrt{A} r , \quad k_j = \frac{2\pi}{L} n_j ; \quad n_j = \pm 1, \pm 2, \pm 3 , \ldots
\]

Where, the parameters as \( m_\nu, \varepsilon_\nu, \mu_\nu \) are the neutron mass, energy and AMM \( \left( \mu_\mu = g_\mu \mu_N \right) \); \( g_\mu = -1.91 \) is Lande factor and \( \mu_N = e/2m_\mu = 5.05 \times 10^{-24} \text{ erg} \cdot \text{G}^{-1} \) is nuclear magneton, respectively. \( F_1 \left( \alpha, \beta, x \right) \) is Confluent Hyper-geometric function [29]. The neutron spin coefficients are

\[
C_{1n} = 1 - s_n , \quad C_{2n} = 1 + s_n . \tag{15}
\]

The neutron spin value \( s_n \left( s_n = \pm 1 \right) \) classifies the neutron states with respect to the spin projection to \( z \) direction \( \left( s_n = + 1 \right) \) corresponds to the spin orientation parallel to the magnetic field \( \vec{B} \). From Eq. (14) and properties of Hyper-geometric function, the energy spectrum for neutron is obtained as follows:

\[
\varepsilon_{n,l} = \frac{k_j^2}{2m_n} + s_n \mu_b \frac{m_n \mu_b^2 a^2}{2 \left( n + l + 0.5 \right)^2} . \tag{16}
\]

This relation is in good agreement with the NU
method [30].

Charged fermions
The Pauli equation for charged fermion taking into account interaction of AMM particles with external constant magnetic field $\vec{B}$ is as follow:

$$\epsilon \psi_{1,2}(r) = \left[ \frac{1}{2m_e} \left( \vec{p} + ea \right)^2 - \left( \mu + \frac{e}{2m_e} \right) B \right] \psi_{1,2}(r).$$  \hspace{1cm} (17)

Solving the Eq. (17) leads to:

$$\psi_{1,2}(r,t) = \frac{1}{L} e^{-\mathcal{H}(p_{1,2},p_{\alpha},\vec{r})} U_{\alpha} (\eta) \left( C_{1e} \right)^{C_{1e}};$$ \hspace{1cm} (18)

$$\eta = \sqrt{2} \mathcal{L} + \frac{p_{r,1}}{\sqrt{2} \mathcal{L}}, \hspace{0.5cm} C_{1e} = 1-s_{e}, \hspace{0.5cm} C_{2e} = 1+s_{e}, \hspace{0.5cm} s_{e} = \pm 1.$$ \hspace{1cm} (19)

Where $U_{\alpha} (\eta)$ is Hermit function.

$$U_{\alpha} (\eta) = \left( -1 \right)^{\mu} \left[ \sqrt{2 \mathcal{L}} \right]^{\mathcal{L}} e^{-\frac{d}{2m_e} \mathcal{L}} \left( \frac{d}{m_e} \right)^{\mathcal{L}} e^{\frac{\alpha}{2m_e} \mathcal{L}} U_{\alpha} (\eta).$$ \hspace{1cm} (20)

The energy spectrum of charged fermion (i.e. electron) is calculated as:

$$e_{e}^{a} = \frac{p_{r,1}^{2}}{2m_e} - \mu_{s_{e}} B + \left( 2n+1 \right) \frac{\gamma}{m_{p}} \frac{eB}{\gamma}.$$ \hspace{1cm} (21)

Here, we suppose that parameter $\alpha$ in Eq. (5) is small. We calculate the contribution of the perturbation term $(\alpha / r)$. Therefore, using the time independent perturbation theory, the electron wave function $\psi_{e}^{(1)} (x, t)$ can be written as follows:

$$\psi_{e}^{(1)} (x, t) = \sum_{a} \sum_{\alpha} \frac{m_{p} H_{n}^{(0)}}{2\mathcal{L} L(n-k)} \exp \left( -i \mathcal{L} t \right) \exp \left( i \left( p_{r,1} x + p_{r,2} y + p_{r,3} z \right) \right) U_{\alpha} (\eta) \left( C_{1e} \right)^{C_{2e}};$$ \hspace{1cm} (22)

$$H^{(e)} = \left( \frac{1}{2m_{p}} \right) \frac{e^{2} \mathcal{L}}{2m_{p}} \mathcal{L} \left( \alpha^{2} \right) \mathcal{L} \left( \mu \right) \mathcal{L} \left( \frac{e}{2m_{e}} \right) B.$$ \hspace{1cm} (23)

Where $\psi_{e}^{(1)} (x, t)$, $\epsilon_{e}$, $m_{e}$, $p_{r,1}$, $p_{r,2}$ and $H^{(e)}$ are the electron wave function of one order correction, energy, mass, momentum components and energy correction under the influence magnetic field, respectively. The proton wave function $\psi_{p}^{(1)} (\vec{x}, t)$ can be expressed in a similar form

$$\psi_{p}^{(1)} (\vec{x}, t) = \sum_{a} \sum_{\alpha} \frac{m_{p} H_{n}^{(0)}}{2\mathcal{L} L(m-k)} \exp \left( -i \mathcal{L} t \right) \exp \left( i \left( p_{r,1} x + p_{r,2} y + p_{r,3} z \right) \right) U_{\alpha} (\eta) \left( C_{1p} \right)^{C_{2p}};$$ \hspace{1cm} (24)

$$\eta^{'} = \sqrt{2} \mathcal{L} x + \frac{p_{r,1}}{\sqrt{2} \mathcal{L}}, \hspace{0.5cm} C_{1p} = 1-s_{p}, \hspace{0.5cm} C_{2p} = 1+s_{p}, \hspace{0.5cm} S_{p} = \pm 1.$$ \hspace{1cm} (25)

The proton parameters are similar to electron ones. The energy spectra of ground state $(n = 0 \text{ or } m = 0)$ is:

$$\langle H \rangle_{n=0} = 4\alpha \sqrt{2} \mathcal{L} \gamma + \beta \gamma, \hspace{0.5cm} \alpha = a \left( \mu + \frac{e}{2m_{e}} \right), \hspace{0.5cm} \beta = \frac{e^2 \mathcal{L}}{2m_{e}}, \hspace{0.5cm} \gamma = \frac{eB}{2}.$$ \hspace{1cm} (26)

Also for excited state $(n = 1 \text{ or } m = 1)$, we have:

$$\langle H \rangle_{n=1} = \frac{13}{2} \sqrt{2} \mathcal{L} \gamma + \frac{47}{4} \frac{e^2 \mathcal{L}}{2m_{e}}, \hspace{0.5cm} \xi = a \left( \mu + \frac{e}{2m_{e}} \right);$$ \hspace{1cm} (27)

Here $m_{p}$, $\mu$ and $e$ are the mass of charged fermions, AMM and charge fermions, respectively.

The general form of cross-section in the non-relativistic limit
In our calculations, we have obtained the exact solution of the Dirac-Pauli equation for neutron in the presence of magnetic field. Also, the non-exact solutions of Dirac-Pauli equation for electron and proton in the non-relativistic limit are obtained using perturbation theory in the presence of non-uniform magnetic field with cylindrical symmetry. Without loss of generality, the cylindrical symmetry magnetic field $\vec{B}$ is taken in z-direction. Substituting these wave functions in Eq. (9) and after some algebra we have:

$$\int \int \int \exp \left( i \left( \epsilon_{e} + \epsilon_{p} - \epsilon_{e} - \epsilon_{p} \right) \right) \exp \left( i \left( p_{r,1} x + p_{r,2} y + p_{r,3} z \right) \right) \exp \left( i \left( \epsilon_{e} - \epsilon_{p} - \epsilon_{e} + \epsilon_{p} \right) \right) dxdydzdt.$$ \hspace{1cm} (28)

Integrating over the coordinate $x$ in the matrix element of beta-decay, we get:

$$M = \frac{2G_{F}}{\sqrt{2}} \int \int \Lambda (\vec{x}, t) \left( C_{1e} C_{n} + C_{2e} C_{n} \right) \left( C_{1p} f_{e} + C_{2p} f_{e} \right)$$ \hspace{1cm} (29)

This equation can be rewritten as:

$$M = \frac{2G_{F}}{\sqrt{2}} \int \Lambda (\vec{x}, t) C dxdydzdt.$$ \hspace{1cm} (30)

Where $\Lambda (\vec{x}, t)$ is defined as:

$$\Lambda (\vec{x}, t) = \frac{1}{2L} \sum_{m,n,k} \frac{m_{p} H_{n}^{(0)}}{4L (m-k) (n-k)} F_{e} (\vec{x}) \left( \psi_{e}^{(1)} \right) \psi_{p}^{(1)} (\eta) dxdydzdt.$$ \hspace{1cm} (31)

Substituting the Eq. (31) into Eq. (30), we obtain
The integration over the electron momentum components, $p_{\nu}$ and $p_{\nu}$, are performed using the two delta functions \( \delta(p_{\nu} + p_{\nu} - p_{\nu} - \chi_{\nu}) \) and \( \delta(p_{\nu} + p_{\nu} - p_{\nu} - \chi_{\nu}) \). After these integrations we get the laws of conservation for the two momentum components, that is \( p_{\nu} = p_{\nu} - \chi_{\nu} - p_{\nu} \), \( p_{\nu} = p_{\nu} - \chi_{\nu} - p_{\nu} \). Finally, we have obtained the cross section of the beta decay of polarized neutron in a cylindrical symmetry external magnetic field in the non-relativistic limit as:

\[
\sigma = \frac{16(\pi \gamma)^{2}}{2m_{\nu}} \sum \sum H_{00}^{1} H_{00}^{2} H_{00}^{3} H_{00}^{4} m_{\nu}^{2} \frac{G_{F}^{2}}{4\pi} \left( \frac{\gamma}{m_{\nu}} \right) \delta(\epsilon, \epsilon - \epsilon)
\]

This relation is the general form of cross-section in the non-relativistic limit. We can obtain cross-section of neutron decay in ground state by inserting $k = q = n = m = 1$ into Eq. (39).

### Discussion

In this survey, we have developed the beta-decay of polarized neutron in the presence of strong external uninform field (variable with cylindrical symmetry). It is known a strong magnetic field can lead to beta-decay in neutron stars. In order to the description of the energy spectrum and wave functions in the URCA process, we have employed the Dirac-Pauli equation.

The Dirac-Pauli equation has been solved for neutral and charged fermions with account of AMM in the presence of a strong magnetic field with a cylindrical symmetry. Here, we have used the perturbation method for solving the Dirac-Pauli equation. The energy spectrum for massive neutral fermions in the non-relativistic limit is calculated and compared with the Nikiforov-Uvarov method (NU) [30]. The results are in good agreement together. This means that the wave functions for the non-relativistic fermions are valid. We know that the neutrinos move with the speed of light. They have not interaction with magnetic field, so we have used the relativistic wave functions of neutrinos in the presence of constant magnetic field [21] in our calculations. These wave functions have been used in calculating the matrix element \( \langle \hat{M} \rangle \) of the process.

The amplitude of the bound-state decay process \( \langle \hat{M} \rangle \)
has been formulated by using the four-fermion Lagrangian within the framework of the standard model of weak interactions.

These calculations from the perspective of nuclear astrophysics can be important in estimating the neutron decay cross section. We have obtained the general form for calculating the cross section of beta decay in the non-relativistic limit. This relation can be important from the perspective of nuclear astrophysics. Since, experimental astrophysics data's satisfy the existence of strong cylindrical symmetric magnetic field on the surface of forced magnetized neutron stars, our calculations and results are valid, in point of astrophysics observation.

References
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