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# (2n+1)-Weak Module Amenability of Triangular Banach Algebras on Inverse Semigroup Algebras

E. Nasrabadi[1*], M. Ramezanpour[2], A. Aasaraai[3]

[1] *Department of Mathematic, Faculty of Mathematic Science and Statistics, University of Birjand, Birjand, Islamic Republic of Iran*
[2] *School of Mathematics and Computer Science, Damghan University, Damghan, Islamic Republic of Iran*
[3] *Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Islamic Republic of Iran*

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## Abstract

Let $S$ be a commutative (not necessary unital) inverse semigroup with the set of idempotents $E$ then $\ell^1(S)$ is a commutative Banach $\ell^1(E)$-module with canonical actions. Recently, it is shown that the triangular Banach algebra

$$\mathcal{T} = \begin{bmatrix} \ell^1(S) & M \\ & \ell^1(S) \end{bmatrix}$$

is $(n)$-weakly $\ell^1(E)$-module amenable, provided that $M = \ell^1(S)$ and $S$ is unital or $E$ satisfies condition $D_k$ for some $k \in \mathbb{N}$. In this paper, we show that $\mathcal{T}$ is $(2n+1)$-weakly $\ell^1(E)$-module amenable, without any additional conditions on $S$ and $E$, if $M$ is a certain quotient space of $\ell^1(S)$.

**Keywords:** Inverse semigroup; Triangular Banach algebra; First module cohomology group; Weak module amenability; (n)-weak module amenability.

## Introduction

Let $A$ and $B$ be Banach algebras, and $M$ be a Banach $A, B$-module, that means, $M$ is a left Banach $A$-module and right Banach $B$-module. Then

$$\mathcal{T} = Tri(A, B, M) = \left\{ \begin{bmatrix} a & m \\ & b \end{bmatrix} : a \in A, b \in B, m \in M \right\}.$$

considered with the usual operations associated with $2 \times 2$ matrices and the Banach space norm $\left\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \right\| = \|a\| + \|m\| = \|b\|$, becomes a Banach algebra which is called a triangular Banach algebra. This class of Banach algebras was studied by Forrest and Marcoux in [1]. They have researched the permanent weak amenability of a type of Banach algebra known as triangular Banach algebras in [2]. They consider the cases where $A$ and $B$ have units and $M$ is unital $A, B$-module, and show that the weak amenability of $\mathcal{T}$ is tantamount to the weak amenability of the corner Banach algebras $A$ and $B$.

The concept of weak module amenability for Banach, was presented in [3] and was shown for commutative inverse semigroup $S$, the semigroup algebra $\ell^1(S)$ is always weakly amenable as a module over the semigroup algebra $\ell^1(E)$ of its subsemigroup $E$ of idempotents. Also, Bodaghi *et al*, in [4] researched permanent weak module amenability of $\ell^1(S)$. The first author and Pourabbas in [5] and Bodaghi and Jabbari in [6] extended this result and proved that $\ell^1(S)$ is

* Corresponding author: Tel: +985632202078; Email: nasrabadi@birjand.ac.ir

$(2n+1)$-weakly module amenable and $(n)$-weakly module amenable ($n \in \mathbb{N}$) as an $\ell^1(E)$-module, respectively.

In [7], the first author and Pourabbas have studied the weak module amenability of triangular Banach algebras. They confirmed Forrest and Marcoux's results correctly, for weak module amenability instead of weak amenability (See also [8]). As a consequence, the proved that, the triangular Banach algebra $\mathrm{Tri}(\ell^1(S), \ell^1(S), \ell^1(S))$ is weakly $\ell^1(E)$-module amenable, when $S$ is commutative and $\ell^1(S)$ is as an $\ell^1(E)$-module with the following actions:

$\delta_s \cdot \delta_e = \delta_e \cdot \delta_s = \delta_e * \delta_s = \delta_{es} \qquad (s \in S, e \in E)$,

where $\delta_s$ and $\delta_e$ are the point mass at $s \in S$ and $e \in E$, respectively. This result was improved by Bodaghi and Jabbari in [6]. Indeed, they showed that $\mathrm{Tri}(\ell^1(S), \ell^1(S), \ell^1(S))$ is $(n)$-weakly $\ell^1(E)$-module amenable ($n \in \mathbb{N}$), when the idempotent set $E$ satisfies condition $D_k$ for some $k \in \mathbb{N}$.

Let $S$ be a discrete, commutative and not necessarily unital inverse semigroup and $E$ be the set of its idempotent elements. Let $M_0$ be the closed linear span of $\{\delta_{es} - \delta_s : e \epsilon E,\ s \epsilon S\}$ in $\ell^1(S)$ and set $M \coloneqq \ell^1(S)/M_0$. In this paper, we consider $\mathcal{T} = \mathrm{Tri}(\ell^1(S), M, \ell^1(S))$ as an $\mathfrak{T}$-module and $\mathfrak{T}'$-module and show that the first order module cohomology group of $\mathcal{T}$ with coefficients in $\mathcal{T}^{(2n+1)}$ is trivial. In other words, we show that $\mathcal{T}$ is $(2n+1)$-weak $\mathfrak{T}$-module amenable and $(2n+1)$-weak $\mathfrak{T}'$-module amenable, without any additional conditions on $S$ and $E$.

## Preliminaries and Notations

Let $A$ and $\mathfrak{A}$ be Banach algebras and $A$ is a Banach $\mathfrak{A}$-module with compatible actions. Let $X$ to be a Banach space and both $A$-module and $\mathfrak{A}$-module, with compatible actions (for more details see [3], [4], [5], [6], [7], [8] and [9]). In this case we call X is a Banach $\mathfrak{A}$-$A$-module. If $\alpha \cdot x = x \cdot \alpha$ $(a \cdot x = x \cdot a)$ $\quad (\alpha \in \mathfrak{A}, a \in A, x \in X)$, then we call $X$ is a (bi-commutative) commutative Banach $\mathfrak{A}$-A-module.

If $X$ is a (commutative) Banach $\mathfrak{A}$-A-module, then so is $X^*$, when the action $A$ and $\mathfrak{A}$ on $X^*$ are considered by

$(\alpha \cdot f)(x) = f(x \cdot \alpha), \quad (a \cdot f)(x) = f(x \cdot a) \quad (\alpha \in \mathfrak{A}, a \in A, f \in X^*,\ x \in X)$,

and the same for the right actions. Therefore, if $A$ is a commutative Banach $\mathfrak{A}$-module, then $A$ and the its dual space are also commutative Banach $\mathfrak{A}$-$A$-modules.

An $\mathfrak{A}$-module map $D: A \longrightarrow X$ is called an $\mathfrak{A}$-module derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A).$$

Note that, $D: A \longrightarrow X$ is bounded if there exists a constant $M > 0$ such that $\|D(a)\| \leq M \|a\|$, for each $a \in A$. Although $D$ may be nonlinear, but because $D$ preserves subtraction, its boundedness implies its norm continuity.

When $X$ is a commutative Banach $\mathfrak{A}$-$A$-module, any $x \in X$ defines an inner $\mathfrak{A}$-module derivation as follows

$$\boldsymbol{ad}_x(a) = a \cdot x - x \cdot a \qquad (a \in A),$$

If $X$ is a bi-commutative Banach $\mathfrak{A}$-$A$-module, then each inner $\mathfrak{A}$-module derivation is zero. We application the notation $\mathcal{Z}^1_{\mathfrak{A}}(A, X)$ for the set of all $\mathfrak{A}$-module derivations $D: A \longrightarrow X$ and $\mathcal{B}^1_{\mathfrak{A}}(A, X)$for those which are inner. The first order $\mathfrak{A}$-module cohomology group with coefficients in $X$ is denoted by $\mathcal{H}^1_{\mathfrak{A}}(A, X)$ which is the quotient group $\mathcal{Z}^1_{\mathfrak{A}}(A, X)/\mathcal{B}^1_{\mathfrak{A}}(A, X)$.

**Definition 1.1.** An $\mathfrak{A}$-module amenable Banach algebra is a Banach algebra $A$ such that, $\mathcal{H}^1_{\mathfrak{A}}(\mathrm{A}, X^*) = 0$, for any commutative Banach $\mathfrak{A}$-A-module $X$.

**Definition 1.2.** A weak $\mathfrak{A}$-module amenable Banach algebra is a Banach algebra $A$ such that, $\mathcal{H}^1_{\mathfrak{A}}(A, A^*) = 0$.

Let $\mathfrak{A}$, $A$ and B be Banach algebras such that both $A$ and $B$ are Banach $\mathfrak{A}$-modules with compatible actions and let $M$ is a left Banach $A$-module and a right Banach $B$-module (That being said $M$ is a Banach $A, B$-module). Furthermore, let $M$ be commutative Banach $\mathfrak{A}$-$A$-module and commutative Banach $\mathfrak{A}$-$B$-module (See [6], [7] and [8]). Let

$$\mathcal{T} = Tri\,(A, B, M) = \left\{\begin{bmatrix} a & m \\ & b \end{bmatrix} : a \in A, b \in B, m \in M\right\}$$

considered with the usual $2 \times 2$ matrix addition and formal multiplication and the norm $\left\|\begin{bmatrix} a & m \\ & b \end{bmatrix}\right\| = \|a\|_A + \|b\|_B + \|m\|_M$ is a Banach algebra. We call this algebra the triangular Banach algebra. Since, as a Banach space, $\mathcal{T}$ is isomorphic to the $\ell^1$-sum of $A$, B and $M$, it is clear that $\mathcal{T}^* \simeq A^* \oplus_{\ell^\infty} M^* \oplus_{\ell^\infty} B^* = \begin{bmatrix} A^* & M^* \\ & B^* \end{bmatrix}$. Suppose that $\boldsymbol{t} = \begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$ and $\psi = \begin{bmatrix} f & g \\ & h \end{bmatrix} \in \mathcal{T}^*$. Then the action of $\mathcal{T}^*$ upon $\mathcal{T}$ is give by $\psi(\boldsymbol{t}) = f(a) + g(m) + h(b)$. With a little calculation we understand that, module action $\mathcal{T}$ on $\mathcal{T}^*$are as follows:

$$\boldsymbol{t} \cdot \psi = \left(\begin{bmatrix} a \cdot f + m \cdot g & b \cdot g \\ & b \cdot g \end{bmatrix}\right) \quad \text{and} \quad \psi \cdot \boldsymbol{t} = \left(\begin{bmatrix} f \cdot a & g \cdot a \\ & g \cdot m + h \cdot b \end{bmatrix}\right) \qquad (1)$$

Let $\mathfrak{A}$ be a Banach algebra. We define

$$\mathfrak{T} = \left\{\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} : \alpha \in \mathfrak{A}\right\} \simeq \mathfrak{A}.$$

That is, $\mathfrak{T} = \mathrm{Tri}(\mathfrak{A}, \mathfrak{A}, 0)$, so $\mathfrak{T}$ is a Banach algebra. The triangular algebra $\mathcal{T}$ with the usual $2 \times 2$ matrix product is a $\mathfrak{T}$-module, but is not commutative $\mathfrak{T}$-module and so we need additional conditions. Throughout we will drop the sign $\cdot$ to show the module actions.

**Definition 1.3.** A discrete semigroup $S$ is called an inverse semigroup if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. We call $e \in S$ an idempotent if $e = e^* = e^2$. The set of idempotent elements of $S$ is denoted by $E$. Throughout this paper, we consider $S$ is a commutative inverse semigroup with idempotent set $E$. In this case $\ell^1(S)$ is a commutative Banach $\ell^1(E)$-module with the following actions

$$\delta_s \cdot \delta_e = \delta_e \cdot \delta_s = \delta_e * \delta_s = \delta_{es}, \qquad (2)$$

where $\delta_s$ and $\delta_e$ are the point mass at $s \in S$ and $e \in E$, respectively. Also, throughout this paper, we will assume that that $M_0$ is the closed linear span of $\{\delta_{es} - \delta_s : e \epsilon E,\ s \epsilon S\}$ in $\ell^1(S)$, where $M = \frac{\ell^1(S)}{M_0}$. Suppose

$$\mathcal{T} = \mathrm{Tri}\big(\ell^1(S), M, \ell^1(S)\big) = \left\{\begin{bmatrix} a & \boldsymbol{m} \\ & b \end{bmatrix} : a, b \in \ell^1(S), \boldsymbol{m} \in M\right\},$$

$$\mathfrak{T} = \left\{\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} : \alpha \in \ell^1(E)\right\} \simeq \ell^1(E),$$

and

$$\mathfrak{T}' = \left\{\begin{bmatrix} \alpha & \\ & \beta \end{bmatrix} : \alpha, \beta \in \ell^1(E)\right\} \simeq \ell^1(E) \oplus \ell^1(E).$$

In this case $\mathcal{T}^*$ is a commutative $\mathfrak{T}$-$\mathcal{T}$-module and $\mathfrak{T}'$-$\mathcal{T}$-module with the actions (1). For every $\boldsymbol{m} \in M$ there exists $m \in \ell^1(S)$ such that $\boldsymbol{m} = \rho(m) = [m]$, where $\rho: \ell^1(S) \to M$ is the canonical map. Then $\mathcal{T}$ is a commutative $\mathfrak{T}$-module and $\mathfrak{T}'$-module with the following actions

$$\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \cdot \begin{bmatrix} a & [m] \\ & b \end{bmatrix} = \begin{bmatrix} a & [m] \\ & b \end{bmatrix} \cdot \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} = \begin{bmatrix} \alpha a & [\alpha m] \\ & \alpha b \end{bmatrix}$$

and

$$\begin{bmatrix} \alpha & \\ & \beta \end{bmatrix} \cdot \begin{bmatrix} a & [m] \\ & b \end{bmatrix} = \begin{bmatrix} \alpha a & [\alpha m] \\ & \beta b \end{bmatrix}, \quad \begin{bmatrix} a & [m] \\ & b \end{bmatrix} \cdot \begin{bmatrix} \alpha & \\ & \beta \end{bmatrix} = \begin{bmatrix} a\alpha & [\beta m] \\ & \beta b \end{bmatrix},$$

For all $\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \in \mathfrak{T}$, $\begin{bmatrix} \alpha & \\ & \beta \end{bmatrix} \in \mathfrak{T}'$ and $\begin{bmatrix} a & [m] \\ & b \end{bmatrix} \in \mathcal{T}$.

By induction, on $n \in \mathbb{N} \cup \{0\}$, we find that the $\mathcal{T}$-bimodule actions on $\mathcal{T}^{(2n+1)}$ are formulated as follows:

$$\begin{bmatrix} a & [m] \\ & b \end{bmatrix} \cdot \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} = \begin{bmatrix} a\phi + [m]\varphi & b\varphi \\ & b\psi \end{bmatrix}, \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \cdot \begin{bmatrix} a & [m] \\ & b \end{bmatrix} = \begin{bmatrix} \phi a & \varphi a \\ & \psi b + \varphi[m] \end{bmatrix}$$

for all $\begin{bmatrix} a & [m] \\ & b \end{bmatrix} \in \mathcal{T}$ and $\begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \in \mathcal{T}^{(2n+1)}$.

***(2n+1)-Weak module amenability of $Tri(\ell^1(S), M, \ell^1(S))$ as an $\ell^1(E)$-module***

In this section, we examine the conditions in which $\mathcal{T}$ is $(2n+1)$-weak module amenable as an $\mathfrak{T}$-module. To clarify the relation between $(2n+1)$-weak $\mathfrak{T}$-module amenability $\mathcal{T}$ and $(2n+1)$-weak $\ell^1(E)$-module amenability $\ell^1(S)$, we need the following lemma. The proof of the next lemma is elementary and left to the reader (compare with [7, Lemma 1.1]). Before that, we need to be reminded that the maps $\pi_1, \pi_4 : \mathcal{T}^{(2n+1)} \longrightarrow \ell^1(S)^{(2n+1)}$ are projection maps on arrays. Indeed,

$$\pi_1\left(\begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix}\right) = \phi \quad \text{and} \quad \pi_4\left(\begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix}\right) = \psi \quad \left(\begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \in \mathcal{T}^{(2n+1)}\right).$$

**Lemma 2.1.** Let $D: \mathcal{T} \longrightarrow \mathcal{T}^{(2n+1)}$ be a continuous $\mathfrak{T}$-module derivation and define $D_1$ and $D_4: \ell^1(S) \longrightarrow \ell^1(S)^{(2n+1)}$ by

$$D_1(a) := \pi_1\left(D\left(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\right)\right) \quad \text{and} \quad D_4(b) := \pi_4\left(D\left(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\right)\right). \qquad (3)$$

Then $D_1$ and $D_4$ are $\ell^1(E)$-module derivations. Conversely, if $D_1$ and $D_4 \in \mathcal{Z}^1_{\ell^1(E)}\big(\ell^1(S), \ell^1(S)^{(2n+1)}\big)$ and $D_{14}: \mathcal{T} \longrightarrow \mathcal{T}^{(2n+1)}$ is defined by

$$D_{14}\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) = \begin{bmatrix} D_1(a) & \\ & D_4(b) \end{bmatrix},$$

then $D_{14} \in \mathcal{Z}^1_{\ell^1(E)}\big(\ell^1(S), \ell^1(S)^{(2n+1)}\big)$. Furthermore, $D_{14}$ is inner if and only if $D_1$ and $D_4$ are inner.

**Lemma 2.2.** Let $D: \mathcal{T} \longrightarrow \mathcal{T}^{(2n+1)}$ be a continuous $\mathfrak{T}$-module derivation. Then

$$D\left(\begin{bmatrix} \delta_e & 0 \\ & \delta_e \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix}, \qquad (4)$$

$$D\left(\begin{bmatrix} \delta_e & 0 \\ & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \gamma \\ & 0 \end{bmatrix}, \qquad (5)$$

$$D\left(\begin{bmatrix} 0 & 0 \\ & \delta_e \end{bmatrix}\right) = \begin{bmatrix} 0 & -\gamma \\ & 0 \end{bmatrix}, \qquad (6)$$

for each $e \in E$ and some $\gamma \in M^{(2n+1)}$.

*Proof:* We first prove by induction on $k \in \mathbb{N}$ that

$$\Phi\delta_x = \Phi = \Phi\delta_x \qquad (\Phi \in M^{(k)}, x \in E). \tag{7}$$

We proceed by induction on $k$. For $k = 1$, let $\phi \in M^{(1)} = M^*$, since $M^* = M_0^{\perp}$,

$$(\phi\delta_x - \phi)(\delta_s) = \phi(\delta_{xs} - \delta_s) = 0,$$

for every $s \in E$. From linearity and continuity, we see that (7) works for $k = 1$, since $S$ is commutative. Assume that (7) holds for every $k \geq 2$. Let $\Phi \in M^{(k+1)}$ and $x \in E$, by the induction hypothesis,

$$\Phi\delta_x(\phi) = \Phi(\delta_x\phi) = \Phi(\phi) = \Phi(\phi\delta_x) = \delta_x\Phi(\phi),$$

for every $\phi \in M^{(k)}$. Therefore, $\Phi\delta_x = \Phi$ and (7) is proven. To prove (4), let $e \in E$. Since $\begin{bmatrix}\delta_e & 0\\ & \delta_e\end{bmatrix} = \begin{bmatrix}\delta_e & 0\\ & \delta_e\end{bmatrix}\begin{bmatrix}\delta_e & 0\\ & \delta_e\end{bmatrix}$ and $D$ is an $\mathfrak{T}$-module derivations, we have

$$D\left(\begin{bmatrix}\delta_e & 0\\ & \delta_e\end{bmatrix}\right) = 2D\left(\begin{bmatrix}\delta_e & 0\\ & \delta_e\end{bmatrix}\right).$$

Therefore, $D\left(\begin{bmatrix}\delta_e & 0\\ & \delta_e\end{bmatrix}\right) = \begin{bmatrix}0 & 0\\ & 0\end{bmatrix}$. By Lemma 2.1,

$$D\left(\begin{bmatrix}\delta_e & 0\\ & 0\end{bmatrix}\right) = \begin{bmatrix}D_1(\delta_e) & \gamma_e\\ & \theta_e\end{bmatrix} = \begin{bmatrix}0 & \gamma_e\\ & \theta_e\end{bmatrix},$$ for some $\gamma_e \in M^{(2n+1)}$ and $\theta_e \in \ell^1(S)^{(2n+1)}$. By the observation,

$$D\left(\begin{bmatrix}\delta_e & 0\\ & 0\end{bmatrix}\right) = D\left(\begin{bmatrix}\delta_e & 0\\ & 0\end{bmatrix}\begin{bmatrix}\delta_e & 0\\ & 0\end{bmatrix}\right),$$

it follows immediately that

$$D\left(\begin{bmatrix}\delta_e & 0\\ & 0\end{bmatrix}\right) = \begin{bmatrix}0 & \gamma_e\delta_e\\ & 0\end{bmatrix} = \begin{bmatrix}0 & \gamma_e\\ & 0\end{bmatrix}.$$

Since $D$ is additive,

$$D\left(\begin{bmatrix}0 & 0\\ & \delta_e\end{bmatrix}\right) = D\left(\begin{bmatrix}\delta_e & 0\\ & \delta_e\end{bmatrix} - \begin{bmatrix}\delta_e & 0\\ & 0\end{bmatrix}\right) = \begin{bmatrix}0 & -\gamma_e\\ & 0\end{bmatrix}.$$

To complete the proof, we need to show that $\gamma_e = \gamma_x$, for all $x \in E$. To this end, let $x \in E$. Since $D\left(\begin{bmatrix}\delta_e & 0\\ & \delta_x\end{bmatrix}\right) = D\left(\begin{bmatrix}\delta_e & 0\\ & \delta_x\end{bmatrix}\begin{bmatrix}\delta_e & 0\\ & \delta_x\end{bmatrix}\right)$, so we have

$$\begin{aligned}\begin{bmatrix}0 & \gamma_e - \gamma_x\\ & 0\end{bmatrix} &= \begin{bmatrix}0 & \gamma_e - \gamma_x\\ & 0\end{bmatrix}\begin{bmatrix}\delta_e & 0\\ & \delta_x\end{bmatrix} + \begin{bmatrix}\delta_e & 0\\ & \delta_x\end{bmatrix}\begin{bmatrix}0 & \gamma_e - \gamma_x\\ & 0\end{bmatrix}\\ &= \begin{bmatrix}0 & (\gamma_e - \gamma_x)\delta_e\\ & 0\end{bmatrix} + \begin{bmatrix}0 & \delta_x(\gamma_e - \gamma_x)\\ & 0\end{bmatrix}\\ &\overset{\text{(By)}}{=} \begin{bmatrix}0 & 2(\gamma_e - \gamma_x)\\ & 0\end{bmatrix}.\end{aligned}$$

Therefore, $\gamma_e = \gamma_x$. □

**Lemma 2.3.** Let $D: \mathcal{T} \longrightarrow \mathcal{T}^{(2n+1)}$ be a continuous $\mathfrak{T}$-module derivation and $\gamma \in M^{(2n+1)}$ be as in Lemma 2.2. Then, for every $e \in E$,

$$D\left(\begin{bmatrix}0 & [\delta_e]\\ & 0\end{bmatrix}\right) = \begin{bmatrix}-[\delta_e]\gamma & 0\\ & \gamma[\delta_e]\end{bmatrix}. \tag{8}$$

*Proof:* Let $D\left(\begin{bmatrix}0 & [\delta_e]\\ & 0\end{bmatrix}\right) = \begin{bmatrix}\zeta_e & \eta_e\\ & \xi_e\end{bmatrix} \in \mathcal{T}^{(2n+1)}$. The equality

$$\begin{bmatrix}0 & [\delta_e]\\ & 0\end{bmatrix} = \begin{bmatrix}0 & [\delta_e]\\ & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ & \delta_e\end{bmatrix},$$

implies that

$$D\left(\begin{bmatrix}0 & [\delta_e]\\ & 0\end{bmatrix}\right) = \begin{bmatrix}-[\delta_e]\gamma & 0\\ & \xi_e\delta_e\end{bmatrix}.$$

From this and using $\begin{bmatrix}0 & [\delta_e]\\ & 0\end{bmatrix} = \begin{bmatrix}\delta_e & 0\\ & 0\end{bmatrix}\begin{bmatrix}0 & [\delta_e]\\ & 0\end{bmatrix}$, we conclude that

$$D\left(\begin{bmatrix}0 & [\delta_e]\\ & 0\end{bmatrix}\right) = \begin{bmatrix}-[\delta_e]\gamma & 0\\ & \gamma[\delta_e]\end{bmatrix}. \qquad \square$$

**Lemma 2.4.** Let $D: \mathcal{T} \longrightarrow \mathcal{T}^{(2n+1)}$ be a continuous $\mathfrak{T}$-module derivation. Then there exist $D_1, D_4 \in \mathcal{Z}^1_{\ell^1(E)}\left(\ell^1(S), \ell^1(S)^{(2n+1)}\right)$ and $\gamma \in M^{(2n+1)}$ such that

$$D\left(\begin{bmatrix}\delta_s & 0\\ & 0\end{bmatrix}\right) = \begin{bmatrix}D_1(\delta_s) & \gamma\delta_s\\ & 0\end{bmatrix}, \tag{9}$$

$$D\left(\begin{bmatrix}0 & 0\\ & \delta_s\end{bmatrix}\right) = \begin{bmatrix}0 & -\delta_s\gamma\\ & D_4(\delta_s)\end{bmatrix}, \tag{10}$$

$$D\left(\begin{bmatrix}0 & \delta_s\\ & 0\end{bmatrix}\right) = \begin{bmatrix}-[\delta_s]\gamma & 0\\ & \gamma[\delta_s]\end{bmatrix}, \tag{11}$$

for every $s \in S$.

*Proof:* Let $D_1$ and $D_4$ be as in Lemma 2.1 and $\gamma \in M^{(2n+1)}$ be as Lemma 2.2. For arbitrary $s \in S$, let $D\left(\begin{bmatrix}\delta_s & 0\\ & 0\end{bmatrix}\right) = \begin{bmatrix}D_1(\delta_s) & \gamma_s\\ & \theta_s\end{bmatrix} \in \mathcal{T}^{(2n+1)}$. Since $ss^* \in E$, from (5) and (7), we get

$$\begin{aligned}D\left(\begin{bmatrix}\delta_s & 0\\ & 0\end{bmatrix}\right) &= D\left(\begin{bmatrix}\delta_{ss^*} & 0\\ & 0\end{bmatrix}\begin{bmatrix}\delta_s & 0\\ & 0\end{bmatrix}\right)\\ &= \begin{bmatrix}0 & \gamma\\ & 0\end{bmatrix}\begin{bmatrix}\delta_s & 0\\ & 0\end{bmatrix} + \begin{bmatrix}\delta_{ss^*} & 0\\ & 0\end{bmatrix}\begin{bmatrix}D_1(\delta_s) & \gamma_s\\ & \theta_s\end{bmatrix}\\ &= \begin{bmatrix}0 & \gamma\delta_s\\ & 0\end{bmatrix} + \begin{bmatrix}\delta_{ss^*}D_1(\delta_s) & 0\\ & 0\end{bmatrix}\\ &= \begin{bmatrix}\delta_{ss^*}D_1(\delta_s) & \gamma\delta_s\\ & 0\end{bmatrix}\\ &= \begin{bmatrix}D_1(\delta_s) & \gamma\delta_s\\ & 0\end{bmatrix}.\end{aligned}$$

This proves (9). To prove (10), let $D\left(\begin{bmatrix}0 & 0\\ & \delta_s\end{bmatrix}\right) = \begin{bmatrix}\mu_s & \gamma_s\\ & D_4(\delta_s)\end{bmatrix}$. Using (6) and (7), since $ss^* \in E$, we have

$$\begin{aligned}D\left(\begin{bmatrix}0 & 0\\ & \delta_s\end{bmatrix}\right) &= D\left(\begin{bmatrix}0 & 0\\ & \delta_s\end{bmatrix}\begin{bmatrix}0 & 0\\ & \delta_{s^*s}\end{bmatrix}\right)\\ &= \begin{bmatrix}\mu_s & \gamma_s\\ & D_4(\delta_s)\end{bmatrix}\begin{bmatrix}0 & 0\\ & \delta_{s^*s}\end{bmatrix} +\end{aligned}$$

$$\begin{bmatrix}0 & 0\\ & \delta_s\end{bmatrix}\begin{bmatrix}0 & -\gamma\\ & 0\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ & D_4(\delta_s)\delta_{s^*s} \end{bmatrix} + \begin{bmatrix} 0 & -\delta_s\gamma \\ & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -\delta_s\gamma \\ & D_4(\delta_s) \end{bmatrix}.$$

Finally, by (8) and (10),

$$D\left(\begin{bmatrix} 0 & [\delta_s] \\ & 0 \end{bmatrix}\right) = D\left(\begin{bmatrix} 0 & [\delta_{ss^*}] \\ & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ & \delta_s \end{bmatrix}\right)$$
$$= \begin{bmatrix} -[\delta_{ss^*}]\gamma & 0 \\ & \gamma[\delta_{ss^*}] \end{bmatrix}\begin{bmatrix} 0 & 0 \\ & \delta_s \end{bmatrix} +$$
$$\begin{bmatrix} 0 & [\delta_{ss^*}] \\ & 0 \end{bmatrix}\begin{bmatrix} 0 & -\delta_s\gamma \\ & D_4(\delta_s) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ & \gamma[\delta_{ss^*}]\delta_{ss^*} \end{bmatrix} + \begin{bmatrix} -[\delta_{ss^*}]\delta_s\gamma & 0 \\ & 0 \end{bmatrix} +$$
$$= \begin{bmatrix} -[\delta_s]\gamma & 0 \\ & \gamma[\delta_s] \end{bmatrix}.$$

Therefore (11) is also proved and this complete the proof. □

Now it's time to present the main result of this section which shows that the conclusion of [7, Lemma 1.1 and 1.2] and [6, Proposition 4.2], holds for the triangular Banach algebra $\mathcal{T} = \mathrm{Tri}(\ell^1(S), M, \ell^1(S))$ without any conditions on commutative inverse semigroup *S*.

**Theorem 2.5.** Let $D: \mathcal{T} \longrightarrow \mathcal{T}^{(2n+1)}$ be a map. Then $D \in \mathcal{Z}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)})$ if and only if

$$D\left(\begin{bmatrix} a & [m] \\ & b \end{bmatrix}\right) = \begin{bmatrix} D_1(a) - [m]\gamma & \gamma a - b\gamma \\ & D_4(b) + \gamma[m] \end{bmatrix}, \quad (12)$$

for some $D_1, D_4 \in \mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2n+1)})$ and $\gamma \in M^{(2n+1)}$. Furthermore, $D$ is inner if and only if $D_1$ and $D_4$ are inner.

*Proof:* To start proving, consider $D \in \mathcal{Z}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)})$, then by Lemmas 2.1 and 2.4, there exist $\gamma \in M^{(2n+1)}$ and $D_1, D_4 \in \mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2n+1)})$ such that (12) holds for point masses. Since the linear combinations of point masses are dense in $\ell^1(S)$, it is enough to show that $D$ is linear. To do this, let $s \in S$ and $\lambda \in \mathbb{C}$, then

$$D\left(\begin{bmatrix} \lambda\delta_s & 0 \\ & 0 \end{bmatrix}\right) = D\left(\begin{bmatrix} \lambda\delta_{ss^*} & 0 \\ & \lambda\delta_{ss^*} \end{bmatrix}\begin{bmatrix} \delta_s & 0 \\ & 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} \lambda\delta_{ss^*} & 0 \\ & \lambda\delta_{ss^*} \end{bmatrix} D\left(\begin{bmatrix} \delta_s & 0 \\ & 0 \end{bmatrix}\right)$$
$$= \lambda\begin{bmatrix} \delta_{ss^*} & 0 \\ & \delta_{ss^*} \end{bmatrix} D\left(\begin{bmatrix} \delta_s & 0 \\ & 0 \end{bmatrix}\right)$$
$$= \lambda D\left(\begin{bmatrix} \delta_{ss^*} & 0 \\ & \delta_{ss^*} \end{bmatrix}\begin{bmatrix} \delta_s & 0 \\ & 0 \end{bmatrix}\right)$$
$$= \lambda D\left(\begin{bmatrix} \delta_s & 0 \\ & 0 \end{bmatrix}\right).$$

Similarly, $D$ is linear on other arrays. The converse and the last part follow from Lemma 2.1 and the observation that

$$D = D_{14} - \boldsymbol{ad}_{\begin{bmatrix} 0 & \gamma \\ & 0 \end{bmatrix}}.$$

Indeed, for $\begin{bmatrix} a & [m] \\ & b \end{bmatrix} \in \mathcal{T}$,

$$D\left(\begin{bmatrix} a & [m] \\ & b \end{bmatrix}\right) = \begin{bmatrix} D_1(a) - [m]\gamma & \gamma a - b\gamma \\ & D_4(b) + \gamma[m] \end{bmatrix}$$
$$= D_{14}\left(\begin{bmatrix} a & [m] \\ & b \end{bmatrix}\right) +$$
$$\begin{bmatrix} -[m]\gamma & -b\gamma \\ & 0 \end{bmatrix} - \begin{bmatrix} 0 & \gamma a \\ & \gamma[m] \end{bmatrix}$$
$$= D_{14}\left(\begin{bmatrix} a & [m] \\ & b \end{bmatrix}\right) -$$
$$\boldsymbol{ad}_{\begin{bmatrix} 0 & \gamma \\ & 0 \end{bmatrix}}\left(\begin{bmatrix} a & [m] \\ & b \end{bmatrix}\right).$$ □

Using Theorem 2.5, we have the next result which will allow us to describe the first order module cohomology group $\mathcal{H}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)})$ of $\mathcal{T}$.

**Theorem 2.6.** Let $\mathcal{T} = \mathrm{Tri}(\ell^1(S), M, \ell^1(S))$. Then

$$\mathcal{H}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)}) \simeq \mathcal{H}^1_{\ell^1(\mathrm{E})}(\ell^1(\mathrm{S}), \ell^1(\mathrm{S})^{(2n+1)}) \oplus \mathcal{H}^1_{\ell^1(\mathrm{E})}(\ell^1(\mathrm{S}), \ell^1(\mathrm{S})^{(2n+1)}).$$

*Proof:* Let $D \in \mathcal{Z}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)})$. By Theorem 2.5, there exist $\ell^1(E)$-module derivations $D_1, D_4 \in \mathcal{Z}^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2n+1)})$ and $\gamma \in M^{(2n+1)}$ such that (12) is valid.

Consider the map:

$$\begin{array}{llll} \Gamma: & \mathcal{Z}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)}) & \longrightarrow & \mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2n+1)}) \oplus \mathcal{H}^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2n+1)}) \\ & D & \mapsto & \left(D_1 + \mathcal{B}^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2n+1)}), D_4 + \mathcal{B}^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2n+1)})\right). \end{array}$$

Clearly, $\Gamma$ is additive. Let $D \in \mathcal{Z}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)})$ and $\lambda \in \mathbb{C}$. Since projection maps on arrays are linear, for every $a \in \ell^1(S)$, we have

$$[\lambda D]_1(a) = \pi_1(\lambda D\left(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\right)) =$$
$$\lambda\, \pi_1(D\left(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\right)) = \lambda D_1(a).$$

This shows that $[\lambda D]_1(a) = \lambda D_1$. Similarly, we have $[\lambda D]_4(a) = \lambda D_4$. Thus, $\Gamma$ is linear. That

$\Gamma$ is surjective follows from Lemma 2.1. Next by Theorem 2.5, we get

$$D \in \mathcal{B}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)}) \Leftrightarrow D_{14} \in \mathcal{B}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)})$$
$$\Leftrightarrow D_1, D_4 \in \mathcal{B}^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)^{(2n+1)})$$
$$\Leftrightarrow D \in \operatorname{Ker}\Gamma.$$

Therefore, $\mathcal{B}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)}) = \operatorname{Ker}\Gamma$. By elementary linear algebra theory, we have

$$\mathcal{H}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)}) = \frac{\mathcal{Z}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)})}{\mathcal{B}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)})}$$

$$= \frac{\mathcal{Z}^1_{\mathfrak{X}}(\mathcal{T},\mathcal{T}^{(2n+1)})}{\operatorname{Ker}\Gamma} \simeq \mathcal{H}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)}) \oplus \mathcal{H}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)}).$$ □

***(2n+1)-Weak module amenability of $Tri(\ell^1(S), M, \ell^1(S))$ as an $\ell^1(E) \oplus \ell^1(E)$-module***

In this section, we consider $S$, $M_0$, $M$ and $\mathcal{T}$ as the previous section and we study the first module cohomology groups $\mathcal{H}^1_{\mathfrak{X}'}(\mathcal{T},\mathcal{T}^{(2n+1)})$, where $n \in \mathbb{N}$ and

$$\mathfrak{X}' = \left\{\begin{bmatrix}\alpha & \\ & \beta\end{bmatrix} : \alpha,\beta \in \ell^1(E)\right\} \simeq \ell^1(E) \oplus \ell^1(E).$$

In this case, we get similar results:

**Lemma 3.1.** *Let* $D: \mathcal{T} \longrightarrow \mathcal{T}^{(2n+1)}$ be continuous $\mathfrak{X}'$-module derivation. Then, for every s ∈ S,

$$D\left(\begin{bmatrix}\delta_s & 0\\ & 0\end{bmatrix}\right) = \begin{bmatrix}D_1(\delta_s) & 0\\ & 0\end{bmatrix},$$

$$D\left(\begin{bmatrix}0 & 0\\ & \delta_s\end{bmatrix}\right) = \begin{bmatrix}0 & 0\\ & D_4(s)\end{bmatrix},$$

$$D\left(\begin{bmatrix}0 & \delta_s\\ & 0\end{bmatrix}\right) = \begin{bmatrix}0 & 0\\ & 0\end{bmatrix},$$

and for every $\begin{bmatrix}a & [m]\\ & b\end{bmatrix} \in \mathcal{T}$,

$$D\left(\begin{bmatrix}a & [m]\\ & b\end{bmatrix}\right) = \begin{bmatrix}D_1(a) & 0\\ & D_4(b)\end{bmatrix},$$

where $D_1$ and $D_4$ are defined by (3).

*Proof:* For $s \in S$, let $D\left(\begin{bmatrix}\delta_s & 0\\ & 0\end{bmatrix}\right) = \begin{bmatrix}D_1(\delta_s) & \gamma_s\\ & \theta_s\end{bmatrix}$. Since $D$ is $\mathfrak{X}'$-module map, we have

$$D\left(\begin{bmatrix}\delta_s & 0\\ & 0\end{bmatrix}\right) = D\left(\begin{bmatrix}\delta_{ss^*} & 0\\ & 0\end{bmatrix}\begin{bmatrix}\delta_s & 0\\ & 0\end{bmatrix}\right)$$

$$= \begin{bmatrix}\delta_{ss^*} & 0\\ & 0\end{bmatrix} D\left(\begin{bmatrix}\delta_s & 0\\ & 0\end{bmatrix}\right)$$

$$= \begin{bmatrix}\delta_{ss^*} & 0\\ & 0\end{bmatrix}\begin{bmatrix}D_1(\delta_s) & \gamma_s\\ & \theta_s\end{bmatrix}$$

$$= \begin{bmatrix}\delta_{ss^*}D_1(\delta_s) & 0\\ & 0\end{bmatrix} = \begin{bmatrix}D_1(\delta_s) & 0\\ & 0\end{bmatrix}.$$

So (15) is true. Similarly, we can prove that (16) is true. Now, let $D\left(\begin{bmatrix}0 & [\delta_s]\\ & 0\end{bmatrix}\right) = \begin{bmatrix}\mu''_s & \gamma''_s\\ & \theta''_s\end{bmatrix}$. By reuse from actions (2), we have

$$D\left(\begin{bmatrix}0 & [\delta_s]\\ & 0\end{bmatrix}\right) = D\left(\begin{bmatrix}\delta_{ss^*} & 0\\ & 0\end{bmatrix}\begin{bmatrix}0 & [\delta_s]\\ & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ & \delta_{s^*s}\end{bmatrix}\right)$$

$$= \begin{bmatrix}\delta_{ss^*} & 0\\ & 0\end{bmatrix} D\left(\begin{bmatrix}0 & [\delta_s]\\ & 0\end{bmatrix}\right)\begin{bmatrix}0 & 0\\ & \delta_{s^*s}\end{bmatrix}$$

$$= \begin{bmatrix}\delta_{ss^*} & 0\\ & 0\end{bmatrix}\begin{bmatrix}\mu''_s & \gamma''_s\\ & \theta''_s\end{bmatrix}\begin{bmatrix}0 & 0\\ & \delta_{s^*s}\end{bmatrix}$$

$$= \begin{bmatrix}\delta_{ss^*}\gamma''_s & 0\\ & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ & \delta_{s^*s}\end{bmatrix} = \begin{bmatrix}0 & 0\\ & 0\end{bmatrix}.$$

So (17) is true. Finally, similar to the Lemma 2.4, since $D$ is additive and continuous, on the other hand functions of finite support are dense in $\ell^1$(S) and every function of finite support is linear combination of point masses, (18) is true. □

**Lemma 3.2.** Let $D: \mathcal{T} \longrightarrow \mathcal{T}^{(2n+1)}$ be $\mathfrak{X}'$-module map and $D_1$ and $D_4$ define by (3). Then $D_1, D_4 \in \mathcal{Z}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)})$ if and only if $D \in \mathcal{Z}^1_{\mathfrak{X}'}(\mathcal{T},\mathcal{T}^{(2n+1)})$. Morevere $D_1, D_4 \in \mathcal{B}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)})$ if and only if $D \in \mathcal{B}^1_{\mathfrak{X}'}(\mathcal{T},\mathcal{T}^{(2n+1)})$.

*Proof:* The proof is the same as that of Lemma 2.5. □

**Theorem 3.3.** Let $S$, $\mathcal{T}$ be as the above and $\mathfrak{X}' \simeq \ell^1(E) \oplus \ell^1(E)$. Then

$$\mathcal{H}^1_{\ell^1(E)\oplus\ell^1(E)}(\mathcal{T},\mathcal{T}^{(2n+1)}) \simeq \mathcal{H}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)}) \oplus \mathcal{H}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)}).$$

*Proof:* Let $D \in \mathcal{Z}^1_{\mathfrak{X}'}(\mathcal{T},\mathcal{T}^{(2n+1)})$. By Lemma 3.1, there exist $\ell^1(E)$-module derivations $D_1$ and $D_4$ such that

$$D\left(\begin{bmatrix}a & m\\ & b\end{bmatrix}\right) = \begin{bmatrix}D_1(a) & 0\\ & D_4(b)\end{bmatrix}.$$

Consider the map:

$$\begin{aligned}\Gamma:\ \mathcal{Z}^1_{\mathfrak{X}'}(\mathcal{T},\mathcal{T}^{(2n+1)}) &\longrightarrow \mathcal{H}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)}) \oplus \mathcal{H}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)})\\ D &\mapsto \left(D_1 + \mathcal{B}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)}), D_4 + \mathcal{B}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)})\right).\end{aligned}$$

The same as that proof of Theorem 2.6 we can show that $\Gamma$ is linear and surjective with $\operatorname{Ker}\Gamma = \mathcal{B}^1_{\mathfrak{X}'}(\mathcal{T},\mathcal{T}^{(2n+1)})$ and

$$\mathcal{H}^1_{\mathfrak{X}'}(\mathcal{T},\mathcal{T}^{(2n+1)}) \simeq \mathcal{H}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)}) \oplus \mathcal{H}^1_{\ell^1(E)}(\ell^1(S),\ell^1(S)^{(2n+1)}).$$ □

## Results and Discussion

Our main destination in this section is to show that the first module cohomology groups $\mathcal{H}^1_{\mathfrak{X}}(\mathcal{T},\mathcal{T}^{(2n+1)})$ and $\mathcal{H}^1_{\mathfrak{X}'}(\mathcal{T},\mathcal{T}^{(2n+1)})$ are zero which means, $(2n+1)$-

weak $\mathfrak{T}$-module amenability and $(2n+1)$-weak $\mathfrak{T}'$-module amenability of the triangular Banach algebra $\mathcal{T}$, respectively, without any additional conditions for commutative inverse semigroup $S$.

The first author and Pourabbas in [5, Theorem 2.4], showed that $\ell^1(S)$ is always $(2n+1)$-weakly $\ell^1(E)$-module amenable; see also [3] and [4]. Thus, as a consequence of Theorem 2.6, we have the next result. Compare with [7, Example 2.3] and [6, Example 1].

**Corollary 4.1.** For every commutative inverse semigroup $S$, $\mathcal{H}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)}) = 0$; or in other words $\mathcal{T}$ is $(2n+1)$-weak $\mathfrak{T}$-module amenable.

Finally, by Theorems 2.6 and 3.3, we have the next results:

**Corollary 4.2.** For every commutative inverse semigroup $S$,

$$\mathcal{H}^1_{\mathfrak{T}}(\mathcal{T}, \mathcal{T}^{(2n+1)}) = \mathcal{H}^1_{\mathfrak{T}'}(\mathcal{T}, \mathcal{T}^{(2n+1)}).$$

**Corollary 4.3.** For every commutative inverse semigroup $S$, $\mathcal{H}^1_{\mathfrak{T}'}(\mathcal{T}, \mathcal{T}^{(2n+1)}) = 0$ and so $\mathcal{T}$ is $(2n+1)$-weak $\ell^1(E) \oplus \ell^1(E)$-module amenable.

The question that can naturally be asked at the end is: Under which conditions, $\mathcal{T}$ is permanently weakly module amenable?

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