

HYPERSTRUCTURES ASSOCIATED WITH ORDERED SETS

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ABSTRACT. In this paper some new hyperoperations are introduced in a different context and analyzed. The setting is as follows. In the place of the membership function $\mu : U \rightarrow [0, 1]$, we have a function $\tilde{A}$ from a finite universe H to a totally ordered set $\langle V; \leq \rangle$. Finally one determines a necessary and sufficient condition on a hypergroup $\langle H; \circ \rangle$ such that it may be considered as the join space associated with a fuzzy set $\langle H; \mu \rangle$.

1. Introduction

The concept of algebraic hyperstructures, which is a generalization of that of algebraic structure, started with Marty's paper [17] and then it was developed by many researchers. They used it in different contexts: algebraic functions, rational fractions, noncommutative groups. At the beginning, general aspects of the theory, connections with group theory and different applications in geometry have been studied. In the last decades, many connections between hyperstructures and other branches of mathematics were considered, leading to applications in geometry, hypergraphs, binary relation, combinatorics, artificial intelligence, automata, rough and fuzzy sets. The book [3] is surveys of the theory of algebraic hyperstructures and their applications. After the introduction of the concept of fuzzy set, various connections between fuzzy sets and algebraic structures have been established and studied. In 1971, Rosenfeld defined the fuzzy subgroup of a group, which led to a new area in the abstract algebra and its applications. Ameri and Zahedi in Iran [1] have extended this connection for constructing a new hyperstructure associated with a group G , which under suitable conditions is a hypergroup and moreover a join space. Another topic on the relations between hyperstructures and fuzzy sets, was introduced by P. Corsini in Italy; he associated a join space with a fuzzy set [4] and a fuzzy set with a hypergroupoid. Relations between Fuzzy Sets, Rough Sets and Hyperstructures, had been already considered, see [4], [5], [6], [7], [8], [9], [10], [12], [15], [16], [22]. Now, in this paper some new hyperoperations are introduced in a different context and analyzed.

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2. Main Results

We shall suppose in the following U to be a non empty finite set, $\tilde{A}$ to be a function from U to a totally ordered set $\langle V, \leq \rangle$. We shall denote, for $\forall(x, y) \in U^2$

$\tilde{A}(x) \vee \tilde{A}(y)$ the maximum between $\tilde{A}(x)$ and $\tilde{A}(y)$,

$\tilde{A}(x) \wedge \tilde{A}(y)$ the minimum between $\tilde{A}(x)$ and $\tilde{A}(y)$.

§1. We consider the following hyperoperations $\eta_1, \dots, \eta_4$ and we shall analyze their properties.

Let be $\forall(a, b) \in U^2$

$$\eta_1) \ a \underset{\vee}{\circ} b = \{x \in U \mid \tilde{A}(x) \leq \tilde{A}(a) \vee \tilde{A}(b)\}$$

$$\eta_2) \ a \underset{\vee}{\circ} b = \{y \in U \mid \tilde{A}(y) \geq \tilde{A}(a) \vee \tilde{A}(b)\}$$

$$\eta_3) \ a \underset{\wedge}{\circ} b = \{u \in U \mid \tilde{A}(u) \leq \tilde{A}(a) \wedge \tilde{A}(b)\}$$

$$\eta_4) \ a \underset{\wedge}{\circ} b = \{v \in U \mid \tilde{A}(v) \geq \tilde{A}(a) \wedge \tilde{A}(b)\}$$

Theorem 2.1. (1) *The hypergroupoids (η_1) , (η_2) , (η_3) , (η_4) are associative.*
 (2) *(η_1) , (η_2) , (η_3) , (η_4) are endowed with identities.*
 (3) *(η_1) , (η_4) are hypergroups.*

Proof. (1) We can remark that $\forall(a, b, c) \in U^3$ we have

$$(a \underset{\vee}{\circ} b) \underset{\vee}{\circ} c = \{x \in U \mid \tilde{A}(x) \leq \tilde{A}(a) \vee \tilde{A}(b) \vee \tilde{A}(c)\} = a \underset{\vee}{\circ} (b \underset{\vee}{\circ} c)$$

$$(a \underset{\vee}{\circ} b) \underset{\vee}{\circ} c = \{y \in U \mid \tilde{A}(y) \geq \tilde{A}(a) \vee \tilde{A}(b) \vee \tilde{A}(c)\} = a \underset{\vee}{\circ} (b \underset{\vee}{\circ} c)$$

$$(a \underset{\wedge}{\circ} b) \underset{\wedge}{\circ} c = \{z \in U \mid \tilde{A}(z) \geq \tilde{A}(a) \wedge \tilde{A}(b) \wedge \tilde{A}(c)\} = a \underset{\wedge}{\circ} (b \underset{\wedge}{\circ} c)$$

$$(a \underset{\wedge}{\circ} b) \underset{\wedge}{\circ} c = \{u \in U \mid \tilde{A}(u) \leq \tilde{A}(a) \wedge \tilde{A}(b) \wedge \tilde{A}(c)\} = a \underset{\wedge}{\circ} (b \underset{\wedge}{\circ} c)$$

(2) Set

$$x_0 : \tilde{A}(x_0) = \min\{\tilde{A}(x) \mid x \in U\}$$

$$x_1 : \tilde{A}(x_1) = \max\{\tilde{A}(x) \mid x \in U\}$$

We have clearly that x_0 is an identity for both $\langle U; \underset{\vee}{\circ} \rangle$ and $\langle U; \underset{\wedge}{\circ} \rangle$. x_1 is an identity for both $\langle U; \underset{\vee}{\circ} \rangle$ and $\langle U; \underset{\wedge}{\circ} \rangle$.

(3) $\forall(a, b) \in U$ we have $a \in a \underset{\vee}{\circ} b$ and $a \in a \underset{\wedge}{\circ} b$. □

Theorem 2.2. (1) *U coincides with the set of identities of $\langle U; \underset{\vee}{\circ} \rangle$ and $\langle U; \underset{\wedge}{\circ} \rangle$.*

(2) *The set of identities of $\langle U; \underset{\vee}{\circ} \rangle$ is $\tilde{A}^{-1}\tilde{A}(x_0)$. The set of identities of $\langle U; \underset{\wedge}{\circ} \rangle$ is $\tilde{A}^{-1}\tilde{A}(x_1)$.*

(3) *$\langle U; \underset{\vee}{\circ} \rangle$ and $\langle U; \underset{\wedge}{\circ} \rangle$ are regular reversible hypergroups.*

(4) *$\langle U; \underset{\vee}{\circ} \rangle$ and $\langle U; \underset{\wedge}{\circ} \rangle$ are not hypergroups.*

Proof. 1) Indeed $\forall x \in U, \forall u \in U$, we have

$$\tilde{A}(x) \leq \tilde{A}(x) \vee \tilde{A}(u), \quad \tilde{A}(x) \geq \tilde{A}(x) \wedge \tilde{A}(u)$$

whence $x \in x \overset{\leq}{\underset{\vee}{\circ}} u, x \in x \overset{\geq}{\underset{\wedge}{\circ}} u$.

2) Set $z \in A(x_0), v \in A(x_1), x \in U$. Then

$$\tilde{A}(x) = \tilde{A}(x) \vee \tilde{A}(z) \text{ whence } x \in x \overset{\geq}{\underset{\vee}{\circ}} z$$

$$\tilde{A}(x) = \tilde{A}(x) \wedge \tilde{A}(v) \text{ whence } x \in x \overset{\leq}{\underset{\wedge}{\circ}} v.$$

On the other hand, if $y \notin A(x_0)$ we have $x_0 \notin x_0 \overset{\geq}{\underset{\vee}{\circ}} y$

It follows that y is not an identity for (η_2) . Hence $\tilde{A}^{-1}\tilde{A}(x_0) = E_{\vee}^{\geq}$ = the set of identities of (η_2) . By a similar proof one sees that $\tilde{A}^{-1}\tilde{A}(x_1) = E_{\wedge}^{\leq}$.

3) It follows from (1).

4) $\langle U; \overset{\geq}{\underset{\vee}{\circ}} \rangle, \langle U; \overset{\leq}{\underset{\wedge}{\circ}} \rangle$ are not hypergroups.

Indeed, if $\tilde{A}(b) < \tilde{A}(a)$, x does not exist such that $b \in a \overset{\geq}{\underset{\vee}{\circ}} x$ that is $\tilde{A}(b) \geq \tilde{A}(a) \vee \tilde{A}(x)$.

Similarly, if $\tilde{A}(b) > \tilde{A}(a)$, y does not exist such that $\tilde{A}(b) \leq \tilde{A}(a) \wedge \tilde{A}(y)$, that is $a \in b \overset{\leq}{\underset{\wedge}{\circ}} y$. $\square$

Definition 2.3. I. We call *quasi-join space* a commutative semi-hypergroup which satisfies the condition **(j)**: $a/b \cap c/d \neq \emptyset \implies a \circ d \cap b \circ c \neq \emptyset$.

II. We call *semi-join space* a semi-hypergroup which satisfies the condition **(j)**.

Theorem 2.4. $\langle U; \overset{\leq}{\underset{\vee}{\circ}} \rangle, \langle U; \overset{\geq}{\underset{\wedge}{\circ}} \rangle$ are join spaces, $\langle U; \overset{\geq}{\underset{\vee}{\circ}} \rangle, \langle U; \overset{\leq}{\underset{\wedge}{\circ}} \rangle$ are semi-join spaces.

Proof. Set $\forall (a, b) \in U^2$

$$a \wedge b = a \quad \text{if and only if} \quad A(a) \leq A(b)$$

$$a \wedge b = b \quad \text{if and only if} \quad A(b) \leq A(a)$$

$$a \vee b = a \quad \text{if and only if} \quad A(a) \geq A(b)$$

$$a \vee b = b \quad \text{if and only if} \quad A(b) \geq A(a)$$

Follows $\tilde{A}(a \wedge b) = \tilde{A}(a) \wedge \tilde{A}(b), \tilde{A}(a \vee b) = \tilde{A}(a) \vee \tilde{A}(b)$.

(ε_1) Set $(a, b, c, d) \in U^4, y = a \wedge c$. We have

$$y \leq a \leq a \vee d \quad \text{whence} \quad y \in a \overset{\leq}{\underset{\vee}{\circ}} d$$

$$y \leq c \leq b \vee c \quad \text{whence} \quad y \in b \overset{\leq}{\underset{\vee}{\circ}} c$$

So $a \overset{\leq}{\underset{\vee}{\circ}} d \cap b \overset{\leq}{\underset{\vee}{\circ}} c \neq \emptyset$. Hence $\langle U; \overset{\leq}{\underset{\vee}{\circ}} \rangle$ by (3) Theorem 2.1(3), is a join space.

(ε_4) By a similar proof one sees that $a \vee c \in a \overset{\geq}{\underset{\wedge}{\circ}} d \cap b \overset{\geq}{\underset{\wedge}{\circ}} c$. So, by Theorem 2.1(3), $\langle U; \overset{\geq}{\underset{\wedge}{\circ}} \rangle$ is a join space.

(ε_2) Set $x \in a/b \cap c/d$. Then

$$\begin{aligned} a \in b \overset{\triangleright}{\underset{\vee}{\circ}} x &= \{z \mid \tilde{A}(z) \geq \tilde{A}(b) \vee \tilde{A}(x)\} \text{ and} \\ c \in d \overset{\triangleright}{\underset{\vee}{\circ}} x &= \{z \mid \tilde{A}(z) \geq \tilde{A}(d) \vee \tilde{A}(x)\} \end{aligned}$$

Follows

$$\tilde{A}(a \vee c) = \tilde{A}(a) \vee \tilde{A}(c) \geq \tilde{A}(b) \vee \tilde{A}(d) \vee \tilde{A}(x) \geq \tilde{A}(b) \vee \tilde{A}(d).$$

Hence

$$\begin{aligned} \tilde{A}(a \vee c) &\geq \tilde{A}(a) \vee \tilde{A}(d) \quad \text{whence} \quad a \vee c \in a \overset{\triangleright}{\underset{\vee}{\circ}} d \\ \tilde{A}(a \vee c) &\geq \tilde{A}(b) \vee \tilde{A}(c) \quad \text{whence} \quad a \vee c \in b \overset{\triangleright}{\underset{\vee}{\circ}} c. \end{aligned}$$

Therefore $\langle U; \overset{\triangleright}{\underset{\vee}{\circ}} \rangle$, by Theorem 2.1, is a quasi-join space.

(ε_3) By a similar proof one sees that if in $\langle U; \overset{\triangleleft}{\underset{\wedge}{\circ}} \rangle$, $a/b \cap c/d \neq \emptyset$ then $a \wedge c \in a \overset{\triangleleft}{\underset{\wedge}{\circ}} d \cap b \overset{\triangleleft}{\underset{\wedge}{\circ}} c \neq \emptyset$. Therefore $\langle U; \overset{\triangleleft}{\underset{\wedge}{\circ}} \rangle$ is by Theorem 1.1, a semi-join space. $\square$

§2. We consider now, in a context which generalizes, to the totally ordered set $\langle V; \leq \rangle$, the notion of factor space, new hyperoperations which extend to this setting the hyperoperation $\langle \hat{\circ} \rangle$ defined in §1 of [8].

We shall use the same notations utilized in [14] and in [8].

Let $f : U \rightarrow X(f)$ be a factor of a description frame (U, C, F) , $\alpha \in C$ a concept and $\tilde{A} : U \rightarrow V$ the extension of α . Let η_1^f be the hyperoperation defined in U :

$$a \overset{\triangleleft}{\underset{f\vee}{\circ}} b = \{x \mid \tilde{A}(x) \leq f(\tilde{A})(f(a)) \vee f(\tilde{A})(f(b))\}$$

and let $\eta_2^f, \eta_3^f, \eta_4^f$ be defined analogously.

Since $|U| < \chi_0$, there exist a_0, b_0 such that

$$f(\tilde{A})f(a) = \bigvee_{f(u)=f(a)} \tilde{A}(u) = \tilde{A}(a_0), \quad f(\tilde{A})f(b) = \tilde{A}(b_0)$$

Theorem 2.5. $\langle U; \overset{\triangleleft}{\underset{f\vee}{\circ}} \rangle$, $\langle U; \overset{\triangleright}{\underset{f\vee}{\circ}} \rangle$, $\langle U; \overset{\triangleright}{\underset{f\wedge}{\circ}} \rangle$, $\langle U; \overset{\triangleleft}{\underset{f\wedge}{\circ}} \rangle$ are semi-hypergroups. $\langle U; \overset{\triangleleft}{\underset{f\vee}{\circ}} \rangle$, $\langle U; \overset{\triangleright}{\underset{f\wedge}{\circ}} \rangle$ are hypergroups.

Proof. (δ_1) It follows by the above remark and by Theorem 2.1(1),

$$\forall(a, b, c), (a \overset{\triangleleft}{\underset{f\vee}{\circ}} b) \overset{\triangleleft}{\underset{f\vee}{\circ}} c = \{x \mid \tilde{A}(x) \leq \tilde{A}(a_0) \vee \tilde{A}(b_0) \vee \tilde{A}(c_0)\} = a \overset{\triangleleft}{\underset{f\vee}{\circ}} (b \overset{\triangleleft}{\underset{f\vee}{\circ}} c).$$

By a similar proof one sees that also $\langle U; \overset{\triangleright}{\underset{f\vee}{\circ}} \rangle$, $\langle U; \overset{\triangleleft}{\underset{f\wedge}{\circ}} \rangle$, $\langle U; \overset{\triangleright}{\underset{f\wedge}{\circ}} \rangle$ are associative.

(δ_2) $\forall x$, set x_0 such that $\tilde{A}(x_0) = \max\{\tilde{A}(u) \mid f(u) = f(x)\}$. Set $(a, b) \in U^2$. If we set $x = b$, since $\tilde{A}(b) \leq \tilde{A}(b_0)$, whence $\tilde{A}(b) \leq \tilde{A}(a_0) \vee \tilde{A}(x_0)$, we have $b \in a \overset{\triangleleft}{\underset{f\vee}{\circ}} x$, so $\langle U; \overset{\triangleleft}{\underset{f\vee}{\circ}} \rangle$ is a hypergroup. By a similar proof one sees that $\langle U; \overset{\triangleright}{\underset{f\wedge}{\circ}} \rangle$ is a hypergroup. $\square$

Theorem 2.6. $\langle U; \overset{\triangleleft}{\underset{f\vee}{\circ}} \rangle$, $\langle U; \overset{\triangleright}{\underset{f\wedge}{\circ}} \rangle$ are join spaces.

Proof. Set $y = a \wedge c$ where

$$\begin{aligned} y = a & \quad \text{if and only if} \quad \tilde{A}(a_0) \leq \tilde{A}(c_0) \\ y = c & \quad \text{if and only if} \quad \tilde{A}(c_0) \leq \tilde{A}(a_0) \end{aligned}$$

So

$$\begin{aligned} \text{either} \quad \tilde{A}(y) \leq \tilde{A}(a_0) \leq \tilde{A}(a_0) \vee \tilde{A}(d_0) & \quad \text{whence} \quad y \in a \underset{f \vee}{\overset{<}{\circ}} d \\ \text{or} \quad \tilde{A}(y) \leq \tilde{A}(c_0) \leq \tilde{A}(b_0) \vee \tilde{A}(c_0) & \quad \text{whence} \quad y \in b \underset{f \vee}{\overset{<}{\circ}} c \end{aligned}$$

It follows

$$a \underset{f \vee}{\overset{<}{\circ}} d \cap b \underset{f \vee}{\overset{<}{\circ}} c \neq \emptyset.$$

Therefore, by Theorem 2.1, $\langle U; \underset{f \vee}{\overset{<}{\circ}} \rangle$ is a join space. By a similar proof one sees that $\langle U; \underset{f \wedge}{\overset{>}{\circ}} \rangle$ is a join space. $\square$

Remark 2.7. We have $\forall (x, y) \in U^2$,

- (i) $x \underset{f \vee}{\overset{>}{\circ}} y \subset x \underset{f \vee}{\overset{>}{\circ}} y$ and $x \underset{f \vee}{\overset{<}{\circ}} y \supset x \underset{f \vee}{\overset{<}{\circ}} y$, $x \underset{f \wedge}{\overset{<}{\circ}} y \supset x \underset{f \wedge}{\overset{<}{\circ}} y$ and $x \underset{f \wedge}{\overset{>}{\circ}} y \subset x \underset{f \wedge}{\overset{>}{\circ}} y$.
- (ii) the above inclusions become equalities if f is an injection.

Proof. (i) Indeed, we have $f^{-1}f(\tilde{A}) \supset \tilde{A}$ that is

$$\forall u \in U, \quad \tilde{A}(u) \leq \bigvee_{f(u')=f(u)} \tilde{A}(u').$$

(ii) Follows from Prop. 18 ch. 3 [14]. $\square$

Now, let us suppose the extension of $\alpha \in C$ is not known, but the extension representation of the factor f , that is the function $\tilde{B}(f) : X(f) \rightarrow V$ is known. Then we have the feedback extension of α with respect to f , $\tilde{B}(f) \cdot f : U \rightarrow V$ and we can define the following hyperoperations in U :

$$\begin{aligned} \eta_1'^f \ x \underset{f \vee}{\overset{\leq}{\circ}} y &= \{z \mid \tilde{B}(f)(f(z)) \leq \tilde{B}(f)(f(x)) \vee \tilde{B}(f)(f(y))\} \\ \eta_2'^f \ x \underset{f \wedge}{\overset{\geq}{\circ}} y &= \{z \mid \tilde{B}(f)(f(z)) \geq \tilde{B}(f)(f(x)) \wedge \tilde{B}(f)(f(y))\}. \end{aligned}$$

We can define also in $X(f)$, the following ones: $\forall (a, b) \in X(f) \times X(f)$

$$\begin{aligned} \eta_1''^f \ a \underset{f \vee}{\overset{\leq}{\otimes}} b &= \{c \in X(f) \mid \tilde{B}(f)(c) \leq \tilde{B}(f)(a) \vee \tilde{B}(f)(b)\} \\ \eta_2''^f \ a \underset{f \wedge}{\overset{\geq}{\otimes}} b &= \{c \in X(f) \mid \tilde{B}(f)(c) \geq \tilde{B}(f)(a) \wedge \tilde{B}(f)(b)\} \end{aligned}$$

and analogously to Theorem 2.1, $\eta_3'^f, \eta_4'^f, \eta_3''^f, \eta_4''^f$. As in Theorems ?? one can prove

Theorem 2.8. $\eta_1'^f, \dots, \eta_4'^f, \eta_1''^f, \dots, \eta_4''^f$, are associative. $\langle U; \underset{f \vee}{\overset{\leq}{\circ}} \rangle, \langle U; \underset{f \wedge}{\overset{\geq}{\circ}} \rangle, \langle X(f); \underset{f \vee}{\overset{\leq}{\otimes}} \rangle, \langle X(f); \underset{f \wedge}{\overset{\geq}{\otimes}} \rangle$ are join spaces.

§3. Let (U, C, F) be a description frame and let $\tilde{A}$ be the extension of a concept $\alpha \in C$. Let f, g be in F and let us suppose $f \wedge g = 0$. Let us denote $\forall h \in F$, $x \circ_h y$ the hyperproduct denoted $x \hat{\circ} y$ in §1 [8],

$$x \circ_h y = \left\{ w \mid \tilde{A}(w) \in \left[\bigvee_{h(u)=h(x)} A(u), \bigvee_{h(v)=h(y)} A(v) \right] \right\}.$$

Let us define $\forall u \in U$, $\forall (x, y) \in U^2$

$$\varphi(u) = (f \vee g)^{-1}(\uparrow_f^{f \vee g} f(\tilde{A})) \cap (\uparrow_g^{f \vee g} g(\tilde{A}))(u)$$

(for notations, see Theorem 1.14, ch. 3, [14]),

$$(a_1) \ x \circ_{f \vee g} y = \{z \mid \tilde{A}(z) \in [\varphi(x) \wedge \varphi(y), \varphi(x) \vee \varphi(y)]\}.$$

By [14], Theorem 1.14, ch.3, we have $\varphi(u) = f(\tilde{A})f(u) \wedge g(\tilde{A})g(u)$, so $a_2) \ \varphi(u) =$

$$\left(\bigvee_{f(u')=f(u)} \tilde{A}(u') \right) \wedge \left(\bigvee_{g(v')=g(u)} \tilde{A}(v') \right).$$

Since U is finite, there are x_0, y_0, x_1, y_1 in U such that $f(\tilde{A})(x) = \tilde{A}(x_0)$, $f(\tilde{A})(y) = \tilde{A}(y_0)$, $g(\tilde{A})(x) = \tilde{A}(x_1)$, $g(\tilde{A})(y) = \tilde{A}(y_1)$, that is

$$\begin{aligned} \bigvee_{f(u)=f(x)} A(u) &= A(x_0), & \bigvee_{f(v)=f(y)} A(v) &= A(y_0) \\ \bigvee_{g(s)=g(x)} A(s) &= A(x_1), & \bigvee_{g(t)=g(y)} A(t) &= A(y_1) \end{aligned}$$

It is possible to suppose

$$(i) \quad A(x_0) \leq A(y_0),$$

By $a_1)$ and $a_2)$, we obtain

$$\begin{aligned} x \circ_{f \vee g} y &= \left\{ z \mid \tilde{A}(z) \in \left[\tilde{A}(x_0) \wedge \tilde{A}(x_1) \wedge \tilde{A}(y_0) \wedge \tilde{A}(y_1), \right. \right. \\ &\quad \left. \left. (\tilde{A}(x_0) \wedge \tilde{A}(x_1)) \vee (\tilde{A}(y_0) \wedge \tilde{A}(y_1)) \right] \right\} \end{aligned}$$

We have

Theorem 2.9. $\forall (x, y)$, $x \circ_{f \vee g} y = x \circ_f y$ if and only if

$$(\theta) \quad \forall z \in H, \ \tilde{A}(z_0) \leq \tilde{A}(z_1)$$

Proof. We have indeed, by (θ) ,

$$x \circ_{f \vee g} y = \{z \mid \tilde{A}(z) \in [A(x_0) \wedge A(y_0), A(x_0) \vee A(y_0)]\}$$

It follows

$$x \circ_{f \vee g} y = \{z \mid \tilde{A}(z) \in [A(x_0), A(y_0)]\} = x \circ_f y.$$

On the converse, let us suppose

$$\forall (x, y) \in H^2, \ x \circ_{f \vee g} y = x \circ_f y$$

and by absurd, let us suppose that $\exists z \in H$ such that

$$A(z_1) < A(z_0).$$

Then we have

$$z \circ_f z = \{w \mid \tilde{A}(w) = \tilde{A}(z_0)\}$$

$$\begin{aligned} z \circ_{f \vee g} z &= \{v \mid \tilde{A}(v) \in [\tilde{A}(z_0) \wedge A(z_1), (A(z_0) \wedge A(z_1)) \vee (A(z_0) \wedge A(z_1))]\} = \\ &= \{v \mid \tilde{A}(v) \in [A(z_1), A(z_1)]\} = \{v \mid A(v) = A(z_1)\} \end{aligned}$$

which is different from $z \circ_f z$ since $z_1 \notin z \circ_f z$. $\square$

§4. Let $\langle H; R; \mu \rangle$ be a rough fuzzy set where R is an equivalence on H , $\mu \in [0, 1]^H$, $\forall x \in H$, denote by $R(x)$ the equivalence class of x .

$\forall x \in H$ set

$$\mu_*(x) = \inf\{\mu(s) \mid s \in R(x)\}$$

$$\mu^*(x) = \sup\{\mu(s) \mid s \in R(x)\}.$$

$\forall (x, y) \in H^2$ set

$$x \circ_{R\mu} y = \{z \in H \mid \mu_*(x) \wedge \mu_*(y) \leq \mu_*(z), \mu^*(z) \leq \mu^*(x) \vee \mu^*(y)\}.$$

Theorem 2.10. (i) We have $x \circ_{R\mu} y = \bigcup_{\substack{\mu_*(x) \wedge \mu_*(y) \leq \mu_*(z) \\ \mu^*(x) \vee \mu^*(y) \geq \mu^*(z)}} R(z)$

(ii) $\langle \circ_{R\mu} \rangle$ is associative, whence $\langle H; \circ_{R\mu} \rangle$ is a hypergroup.

Proof. (i) It is enough to remark that if $z \in x \circ_{R\mu} y$, from $z'Rz$, it follows $\mu_*(z') = \mu_*(z)$, $\mu^*(z') = \mu^*(z)$, whence

$$\mu_*(x) \wedge \mu_*(y) \leq \mu_*(z'), \quad \mu^*(z') \leq \mu^*(x) \vee \mu^*(y).$$

(ii) We clearly have

$$(a \circ_{R\mu} b) \circ_{R\mu} c = \bigcup_{\substack{\mu_*(a) \wedge \mu_*(b) \wedge \mu_*(c) \leq \mu_*(z) \\ \mu^*(a) \vee \mu^*(b) \vee \mu^*(c) \geq \mu^*(z)}} R(z) = a \circ_{R\mu} (b \circ_{R\mu} c).$$

Moreover $\langle H; \circ_{R\mu} \rangle$ is a quasi-hypergroup since $\forall (x, y) \in H^2$, $\{x, y\} \subset x \circ_{R\mu} y$. $\square$

Now, let us set:

$$a \circ_{R\mu} b = \{z \mid \mu_*(a) \wedge \mu_*(b) \leq \mu(z) \leq \mu^*(a) \vee \mu^*(b)\}.$$

Remark 2.11. The hyperoperation $\langle \circ_{R\mu} \rangle$ is not generally associative.

It is enough to consider the following example:

Let $H = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be endowed with the equivalence R defined by the partition

$$A_1 = \{1, 2\}, \quad A_2 = \{3, 4\}, \quad A_3 = \{5, 6\}, \quad A_4 = \{7, 8\}$$

and with the membership function μ : $\mu(1) = 0, 2$; $\mu(2) = 0, 4$; $\mu(3) = 0, 1$; $\mu(4) = 0, 3$; $\mu(5) = 0, 05$; $\mu(6) = 0, 15$; $\mu(7) = 0, 07$; $\mu(8) = 0, 24$. We have $\mu_*(1) = 0, 2$; $\mu^*(1) = 0, 4$; $\mu_*(3) = 0, 1$; $\mu^*(3) = 0, 3$; $\mu_*(5) = 0, 05$; $\mu^*(5) = 0, 15$; $\mu_*(7) = 0, 07$; $\mu^*(7) = 0, 24$.

$$\begin{aligned} 5 \underset{R\mu}{\circ} 5 &= \{5, 7, 3, 6\}, & 5 \underset{R\mu}{\circ} 8 &= \{5, 7, 3, 6, 8\}, & 5 \underset{R\mu}{\circ} 3 &= \{5, 3, 7, 8, 6, 4, 1\}, \\ 5 \underset{R\mu}{\circ} 1 &= \{5, 6, 7, 8, 1, 3, 2\}. \end{aligned}$$

It follows

$$(5 \underset{R\mu}{\circ} 5) \underset{R\mu}{\circ} 8 = \{5, 7, 3, 6, 8, 4, 1\}, \quad 5 \underset{R\mu}{\circ} (5 \underset{R\mu}{\circ} 8) = H.$$

Theorem 2.12. $\langle H; \underset{R\mu}{\circ} \rangle$ is an H_v -group (see [23]).

Proof. Indeed, $\forall (x, y, z) \in H^3$ we have $x \underset{R\mu}{\circ} y \supset \{x, y\}$ and moreover $\{x, y, z\} \subset (x \underset{R\mu}{\circ} y) \underset{R\mu}{\circ} z \wedge x \underset{R\mu}{\circ} (y \underset{R\mu}{\circ} z)$. $\square$

§5. Now, let $(I, \leq)$ be a totally ordered set.

Theorem 2.13. Let $\pi = \{A_i\}_{i \in I}$ be a partition of a set H . Let us define

$$(\delta) \quad \forall (x, y) \in A_i^2, \quad x \underset{\pi}{\circ} y = A_i, \quad \text{if } i < j, \quad x \in A_i, \quad y \in A_j, \quad x \underset{\pi}{\circ} y = \bigcup_{i \leq s \leq j} A_s.$$

Then $\langle H; \underset{\pi}{\circ} \rangle$ is a hypergroup.

From Theorem 1.13 we obtain

Theorem 2.14. For every function $\mu : H \rightarrow I$ such that $\forall x \in H, \mu^{-1}\mu(x) = A_{\mu(x)}$, the hypergroupoid defined

$$\forall (x, y) \in H^2, \quad x \underset{\mu}{\circ} y = \{z \mid \mu(x) \wedge \mu(y) \leq \mu(z) \leq \mu(x) \vee \mu(y)\}$$

is a join space which coincides with $\langle H; \circ \rangle$.

So if $I = \mu(H) \subset [0, 1]$, $\langle H; \circ \rangle$ is the join space associated with the fuzzy set $\langle H; \mu \rangle$.

Definition 2.15. We call $\langle H; \circ \rangle$ a I -pr-hypergroup, if a partition $\pi = \{A_i\}_{i \in I}$ of H exists, which satisfies (δ) .

Theorem 2.16. A hypergroup $\langle H; \circ \rangle$ is the join space associated with a fuzzy set if and only if it is an I -pr-hypergroup with $I \subset [0, 1]$.

Proof. It follows by a similar proof to that of Theorem 1.4, §4 [4], from Theorem 1.14 and from Theorem 1.3, §2 [5]. $\square$

Theorem 2.17. Let $\langle H_1; \underset{1}{\circ} \rangle$, $\langle H_2; \underset{2}{\circ} \rangle$ be pr-hypergroups and let $\{A_i\}_{i \in I}$, $\{B_j\}_{j \in J}$ be the partitions of H_1 and H_2 which define the structure of the two hypergroups. Then $H_1 \times H_2$ endowed with the partition $\pi = \{A_i \times B_j\}_{(i,j) \in I \times J}$ is, by (δ) , a pr-hypergroup, where $I \times J$ is the lexicographic product of I and J .

Let $\langle T; \overset{\triangleright}{\underset{\wedge}{\circ}} \rangle, \langle T; \overset{\triangleleft}{\underset{\vee}{\circ}} \rangle$ be hypergroups such that a function exists $f : T \longrightarrow I$ (where $\langle I; \leq \rangle$ is a totally ordered set), so that, setting $\forall \alpha \in I, A_\alpha = \{x \in I \mid x \geq \alpha\}, B_\alpha = \{y \in I \mid y \leq \alpha\}$, we have

$$x \overset{\triangleright}{\underset{\wedge}{\circ}} y = f^{-1}(A_{f(x)}) \cup f^{-1}(A_{f(y)})$$

$$x \overset{\triangleleft}{\underset{\vee}{\circ}} y = f^{-1}(B_{f(x)}) \cap f^{-1}(B_{f(y)})$$

Theorem 2.18. The hyperstructures $\langle T; \overset{\triangleright}{\underset{\wedge}{\circ}} \rangle$ and $\langle T; \overset{\triangleleft}{\underset{\vee}{\circ}} \rangle$ are join spaces.

Proof. The proof is similar to that of Theorem 1.4. By the same way for the following $\square$

Theorem 2.19. Set

$$x \overset{\triangleright}{\underset{\vee}{\circ}} y = f^{-1}(A_{f(x)}) \cap f^{-1}(A_{f(y)}), x \overset{\triangleleft}{\underset{\wedge}{\circ}} y = f^{-1}(B_{f(x)}) \cup f^{-1}(B_{f(y)}).$$

The hyperstructures $\langle H; \overset{\triangleright}{\underset{\vee}{\circ}} \rangle, \langle H; \overset{\triangleleft}{\underset{\wedge}{\circ}} \rangle$ are quasi-join spaces.

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